SLS Note 18/97
November 1997
corr. August 2008

Momentum acceptance and Touschek lifetime

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Contents

Touschek lifetime

Touschek scattering describes a collision of two electrons inside a bunch with transfer of transverse momentum into longitudinal momentum. If the change of longitudinal momentum exeeds the momentum acceptance of the storage ring, both particles get lost. Since the probability of collisions increases with electron density in the bunch, high brightness synchrotron light sources like SLS operating at low emittances and large beam current are usually lifetime limited from Touschek scattering.

Derivations of the Touschek lifetime formula can be found in [B66, D89], using two approximations:

- The transverse particle motion is nonrelativstic.
  Horizontal particle velocity as observed in the laboratory system is simply $v_x = x'\cdot c$, with $x' < 0.1$ mrad for SLS. ($\sigma_{x'}^2 \approx \varepsilon_x/\beta_x$ near a symmetry point, $\varepsilon_x \approx 3$ nm·rad and $\beta_x >$ 0.3 m). In the moving system of the bunch it transforms into $\underline{v}_x = \gamma\cdot v_x$. For SLS at 2.1 GeV ($\gamma = 4200$) we obtain $\underline{v}_x < 0.4$ c, so the nonrelativstic assumption is barely justified.
- The beam is flat, i.e. $z' \ll x'$ everywhere, and the horizontal betatron motion is the only source of Touschek scattering.
  With proposed emittance coupling values of 1% to 10% we get $z' \approx 0.1...0.3\ x'$. So the flat beam assumption too is barely justified.

Both approximations were tested against a complete Monte Carlo simulation and found to affect lifetime results by less than 10% [K93].

The relevance of Touschek scattering may be seen from the following estimate: In nonrelativstic approximation the horizontal momentum in the bunch's system is given by $\underline{p}_x = m_o\cdot\underline{v}_x = m_o\gamma c\ x'$. In the worst case the transverse momentum would be completely transferred into longitudinal momentum: $\underline{p}_s = \underline{p}_x$. Transformation back into the laboratory system multiplies the longitudinal momentum with the relativistic factor γ: $\underline{p}_s = m_o\gamma^2 c x'$. Thus the relative longitudinal momentum as seen in the laboratory would be $p_s/p_o = \gamma\cdot x' \approx 40\%$, exceeding the energy acceptance of the machine which is in the range of 5%.

The Touschek lifetime formula

The Touschek lifetime as the time for the beam intensitiy to decay down to half of its intial value is given by [B66, Z86]:

$$\frac{1}{\tau} = \frac{r_e^2 cq}{8\pi e \gamma^3 \sigma_s} \cdot \frac{1}{C} \cdot \oint_C \frac{F\left(\left(\delta_{acc}(s) \big/ \gamma\sigma_{x'}(s)\right)^2\right)}{\sigma_x(s)\sigma_z(s)\sigma_{x'}(s)\delta_{acc}^2(s)} ds \qquad (1)$$

with

r_e the classical electron radius,

q the bunch charge,

σ_s the rms bunch length, assumed to be constant along the lattice (usually valid for storage rings),

C the machine circumference,

$\sigma_x(s)$, $\sigma_z(s)$ the rms horizontal and vertical beam radii for a flat lattice (no vertical dispersion) given by

$$\sigma_x(s) = \sqrt{\varepsilon_x \beta_x(s) + (\sigma_\delta \eta(s))^2}\,,\;\; \sigma_z(s) = \sqrt{\varepsilon_z \beta_z(s)} \qquad (1a)$$

with

$$\varepsilon_x = \frac{\varepsilon_{xo}}{1+\kappa}\,,\quad \varepsilon_z = \frac{\kappa\varepsilon_{xo}}{1+\kappa} \qquad (1b)$$

the vertical and horizontal emittances for a flat lattice (i.e. no vertical bending magnets), with

ε_{xo} the natural horizontal emittance and

κ the emittance coupling factor;

η, η' the (horizontal) dispersion and its slope and

σ_δ the rms relative momentum spread;

$\sigma_{x'}(s)$ *not* the full rms horizontal beam divergence, but the rms divergence for $x \approx 0$, „since the two interacting electrons are assumed to share the same spatial position“ [K93]. Including dispersion correction $\sigma_{x'}$ is given by [Z86, D89]

$$\sigma_{x'}(s) = \frac{\varepsilon_x}{\sigma_x(s)}\sqrt{1+\frac{H(s)\sigma_\delta^2}{\varepsilon_x}} \qquad (1c$$

with H(s) the chromatic invariant (or dispersion’s emittance) given by

$$H(s) = \gamma_x \eta^2 + 2\alpha_x \eta\, \eta' + \beta_x \eta'^2 \quad , \quad \left(\gamma_x = (1+\alpha_x^2)/\beta_x\right) \qquad (1d)$$

$\delta_{acc}(s)$ the local relative energy acceptance, which can be determined by the RF-system or by the lattice acceptance, and

F(x) a special function defined by $F(x) = \int_0^1 \left(\frac{1}{u} - \frac{1}{2}\ln\frac{1}{u} - 1\right)\cdot e^{-x/u} du$, with the asymptotic expression $F(x) = \ln\left(\frac{E}{x}\right) - 3/2$ for x < 0.01, with E = 0.5772 Euler’s number.

From Eq.1 we see useful Touschek lifetime scaling laws:

$\tau \propto 1/q$ $\qquad$ $\tau \propto \sigma_s$ $\qquad$ $\tau \propto \sqrt{\kappa}$ (if $\kappa \ll 1$, see Eqs.1a, 1b)

The calculation of Touschek lifetime requires determination of momentum acceptance (MA) for every lattice location. The relevant MA is the minimum of the RF MA and the lattice MA. While the RF MA is given by the RF voltage and independent of the location in the lattice, the lattice MA depends on where the scattering event ocurred, and varies along the lattice.

RF momentum acceptance

For a given RF overvoltage V the synchronous phase is given by

$$\sin\varphi_s = U_0 / eV \tag{2}$$

with U_o the energy loss per turn, which is mainly dominated by synchrotron radiation, other contribution like HOM losses in the vacuum chamber may be neglected. The bunchlength is

$$\sigma_s = \sigma_\delta \sqrt{\frac{\alpha C E_o \lambda}{2\pi U_o}\tan\varphi_s} \tag{3}$$

with α the momentum compaction factor of the lattice, E_o the beam energy, and λ the RF wavelength. This formula neglects turbulent bunch lengthening and assumes $U_o \ll E_o$, which is a valid approximation for SLS ($U_o \approx 0.3$ MeV/turn, $E_o = 2.1$ GeV). The RF MA is given by

$$\delta_{acc}^{RF} = \sqrt{\frac{2U_o\ \lambda}{\pi E_o \alpha C}\left(\cot\varphi_s + \varphi_s - \frac{\pi}{2}\right)} \tag{4}$$

This is the maximum half height of the RF bucket at synchronous phase φ_s, for other phases the MA is lower [S70].

Lattice momentum acceptance

Particles undergoing Touschek scattering are particles from the beam core, where the bunch density is high, i.e. their betatron amplitudes are small. After scattering they are still in the beam core, but the relative momenta have suddenly changed to $\pm\delta$. Thus the corresponding off-momentum closed orbit becomes centre of motion for the further betatron motion of the scattered particles. Relative to the new reference orbit the particles suddenly have large betatron motions, so that is is fully justified to neglect the previous betatron amplitudes relative to the on-momentum closed orbit. Subsequently the particles may get lost if their amplitudes exceed the off-momentum geometric or dynamic acceptance of the lattice.

Linear lattice

In a linear lattice the betatron motion is regular for any amplitude and the dynamic acceptance is infinitely large. Thus only the geometric aperture limits a particle's excursion. Since a Touschek scattered particle starts its betatron oscillation at the beam core, i.e. approximately at the origin of phase space, with the dispersive orbit as reference, its betatron amplitude after scattering at lattice location „0“ is given by

$$A_x = \gamma_{xo}(\eta_o\delta)^2 + 2\alpha_{xo}(\eta_o\delta)(\eta_o'\delta) + \beta_{xo}(\eta_0'\delta)^2 = H_o\delta^2\,. \tag{5}$$

Since H(s) varies along the lattice the betatron amplitude depends on where the scattering event took place. At another location „1“ the particles maximum coordinate (we neglect the betatron phase advance between the locations, since the oscillation will persist for many turns) is given by

$$x = \sqrt{A_x\beta_{x1}} + \eta_1\delta = \left(\sqrt{H_o\beta_{x1}} + \eta_1\right)\cdot\delta\ . \tag{6}$$

The particle will be lost if $x > a_{x1}$, with a_{x1} the physical aperture (beam pipe half width) at location „1“. Thus we find for the lattice MA at location „0“

$$\delta^L_{acc}(s_o) = \min_{i=1...N}\left\{\frac{a_{xi}}{\sqrt{H_o\beta_{xi}} + \eta_i}\right\} \tag{7}$$

Nonlinear lattice

In a modern synchrotron light source like SLS the causality {high brightness → low emittance (at limited circumference) → strong focussing → large chromaticity → strong sextupoles} leads to a significantly nonlinearly affected transverse beam optics. In view of calculating the lattice MA we have to face three effects:

- nonlinear variation of twiss parameters α, β, γ with momentum δ,
- higher order dispersion, i.e. nonlinear variation of the closed orbit with δ,
- nonlinear betatron motion: Linear betatron amplitudes (like in Eq.5) are no longer invariants of the motion. However there are other, more complex invariants, since Liouville's theorem still holds (as long as radiation damping is neglected).

The momentum dependant nonlinearities can be derived from a closed orbit finder with subsequent concatenation of the transfermatrix for the periodic solution *along* this orbit, and the off momentum dynamic acceptance, calculated at a point of symmetry, can be propagated along the lattice using the off momentum betafunctions [N97].

The amplitude dependant nonlinearities are more difficult to access, they require higher order maps for description and propagation of the nonlinear betatron amplitudes along the lattice, however these techniques are still perturbative and eventually tracking is used anyway to determine the lattice performance [F92].

Defining lattice MA from the Touschek scattering point of view we take a brute force approach and ask:

> *Will a pair of particles starting at location s with coordinates $(x, x', z, z', \delta) = (0, 0, 0, 0, \pm\delta)$ survive [a sufficient number of turns] or not?*

The particles will follow the nonlinear eigenfigure, no matter how bizarre it may be shaped. No amplitudes are calculated, the loss criterion being simply $|x| > a_{xi}$, (i=1...N).

In practise the maximum positive and negative values δ^+ and δ^- are calculated at every lattice location by binary search for δ. If synchrotron oscillation is to be included implicitly, only the lower absolute value for δ is taken. Two shortcuts help to reduce the computing time required:

- Only bending magnets are able to change H or its nonlinear equivalent. Between the bends it is an invariant of motion, no matter whether there are sextupoles or not. Thus the MA is the same for all locations in a lattice section between bending magnets.
- Symmetric or periodic lattices stay symmetric or periodic even in the nonlinear case.

Touschek relevant effective lattice momentum acceptance

Again taking a strictly Touschek lifetime oriented point of view (since MA is mainly concerned for lifetime issues) we define:

> The Touschek relevant effective lattice momentum acceptance (TRELMA) corresponds to *that* value of RF MA, where the Touschek lifetimes (normalised to bunchlength) calculated using only the RF limitation (assuming infinite lattice MA) and using only the lattice limitation (assuming infinite RF MA) have the same value.

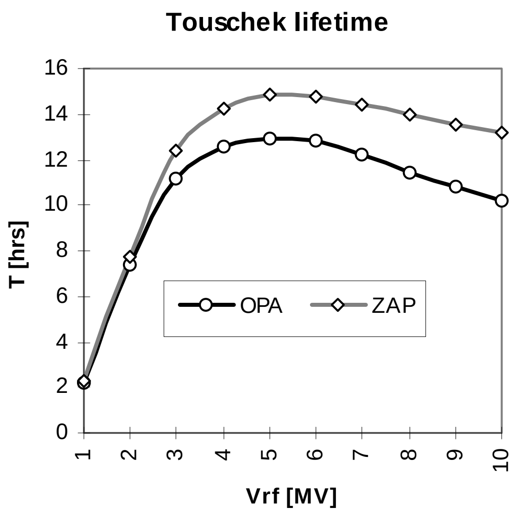
In other words, if the RF voltage is set to give an RF MA as large as the TRELMA both RF and lattice MA contribute equally to the total Touschek lifetime.

Results

The tracking routine for determination of MA at every lattice location was implemented into the code OPA [O97]. Touschek lifetime calculations were done for the SLS storage ring in the two most important modes, for the SLS booster synchrotron and also for the old hexagon shaped SLS storage ring design from 1993 [S93].

Comparison to ZAP

The program ZAP [Z86] is widely used for calculation of collective effects in storage rings. The Touschek module reads a lattice file with betafunctions, dispersions and horizontal aperture for calculation of beam envelopes and MA limitation from the physical aperture. It is also possible to give dynamic aperture data, obtained from some tracking code, at the location of maximum dispersion. These aperture data are propagated linearly using the betafunctions given.

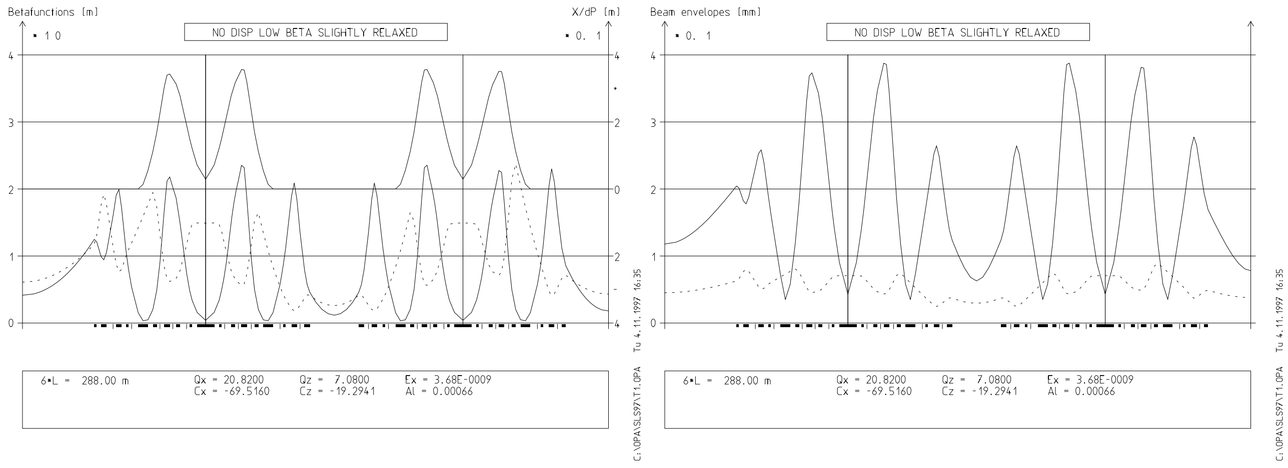


The figure left shows the results for Touschek lifetime as a function of RF voltage for the SLS lattice in D0 mode. Since OPA cannot treat bunchlengthening effects, this option was not activated in ZAP, also intrabeam scattering (anyway irrelelvant for SLS) was neglected.

Parameters: Single bunch charge 5 nCb, or $3.12 \cdot 10^{10}$ particles per bunch, emittance coupling 10%, beam energy 2.1 GeV, vacuum pipe width 65 mm for all elements.

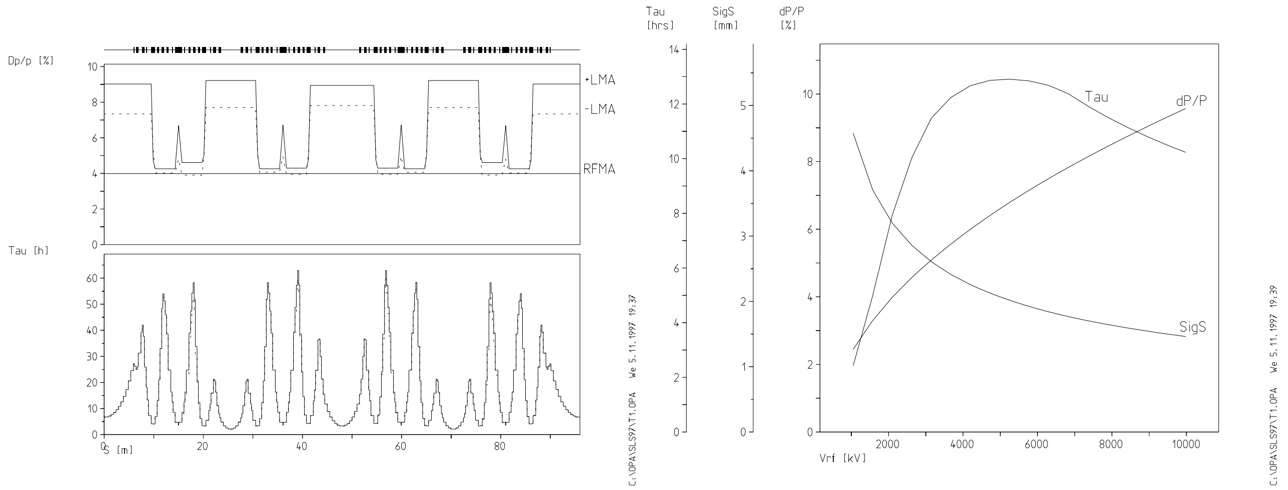
ZAP and OPA agree with their results as long as the Touschek lifetime is determined by the RF MA . When the lattice MA comes into play, the OPA calculation including nonlinear effects gives approx. 15% lower lifetimes than ZAP with the linear model.

SLS storage ring in D0 mode



Beam optics of the SLS storage ring in D0-mode (dispersionfree straight sections

left: Betafunctions, horizontal (solid), vertical (dotted) and dispersion (top line), one sixth of the lattice is shown.
right: Envelopes for 10% coupling



Momentum acceptance and Touschek lifetime results for the SLS storage ring in D0 mode

left: The upper plot shows the MA along one period of the lattice (one third of the ring), with the positive lattice MA(solid line), the negative lattice MA (dotted line) and the RF MA for 2.1 MV overvoltage (straight line). The lower plot shows the Touschek lifetime for each lattice element.

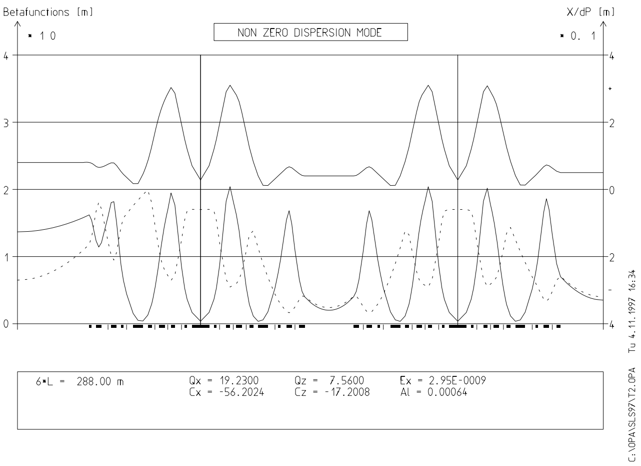
right: Touschek lifetime, RF momentum acceptance and bunchlength as a function of RF voltage.

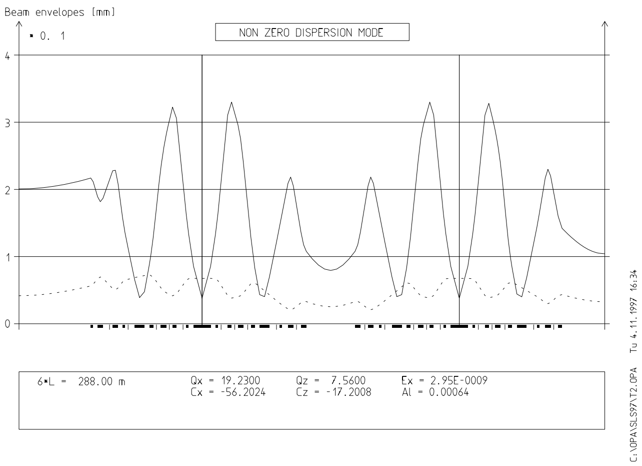
Parameters: Bunch charge q = 5 nCb (= 5.2 mA single bunch current), emittance coupling $\kappa = 10$ %
Beam pipe full width $2a_x = 65$ mm for all elements (vertical is irrelevant here)

Comment: The MA is dominated by the RF at a voltage of 2.1 MV and the Touschek lifetime along the lattice follows mainly the horizontal envelope, the RF MA would reach the TRELMA for V = 3.77 MV. The lattice MA is rather different for the dispersive (4...5%) and for the nondispersive (7...9%) lattice sections, this results in a soft saturation of the Touschek lifetime with increasing RF voltage, because the lattice MA becomes gradually dominant over the RF MA.

SLS (D0 mode, `T1.OPA`) : TRELMA = 5.38 %

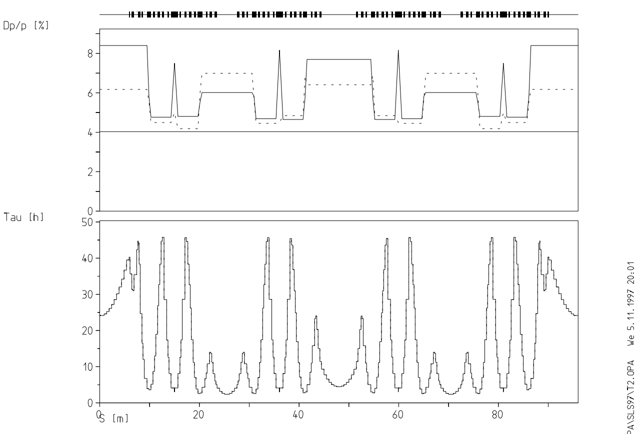
SLS storage ring in D1 mode

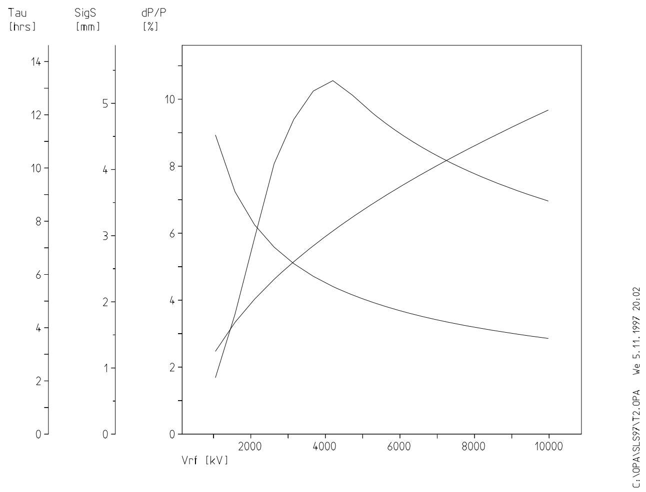




Beam optics of the SLS storage ring in D1-mode (dispersive straight sections

left: Betafunctions, horizontal (solid), vertical (dotted) and dispersion (top line)), one sixth of the lattice is shown.
right: Envelopes for 10% coupling





Momentum acceptance and Touschek lifetime results for the SLS storage ring in D1 mode

left: The upper plot shows the MA along one period of the lattice (one third of the ring), with the positive lattice MA (solid line), the negative lattice MA (dotted line) and the RF MA for 2.1 MV overvoltage (straight line), giving 4% of RF MA.
The lower plot shows the Touschek lifetime for each lattice element.

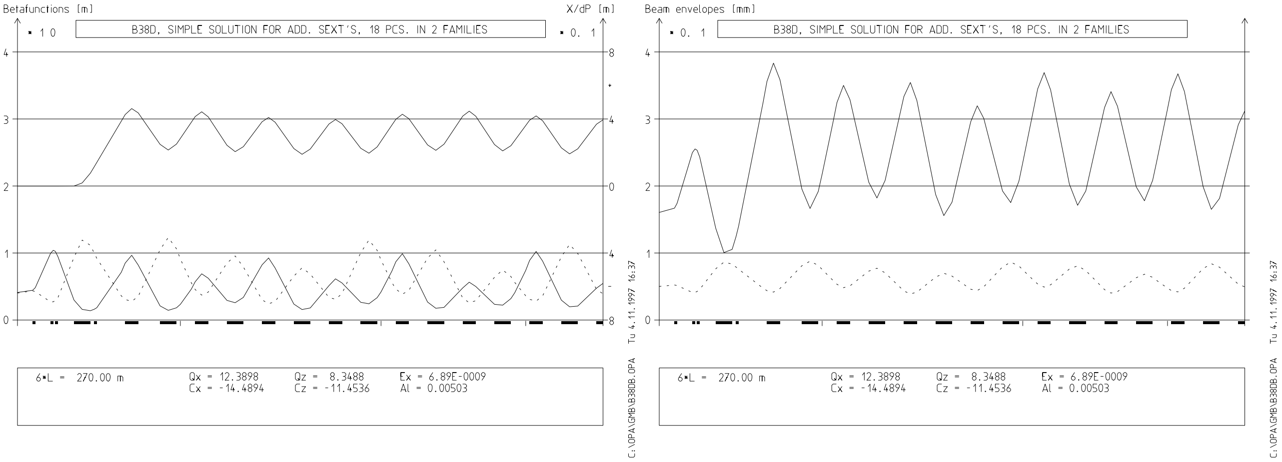
right: Touschek lifetime, RF momentum acceptance and bunchlength as a function of RF voltage.

Parameters: Bunch charge q = 5 nCb (= 5.2 mA single bunch current), emittance coupling κ = 10 %
Beam pipe full width $2a_x$ = 65 mm for all elements (vertical is irrelevant here)

Comment: The MA is dominated by the RF at a voltage of 2.1 MV and the Touschek lifetime along the lattice follows mainly the horizontal envelope.The MA varies less along the lattice than in D0 mode and the lifetime maximum is correspondigly sharper.

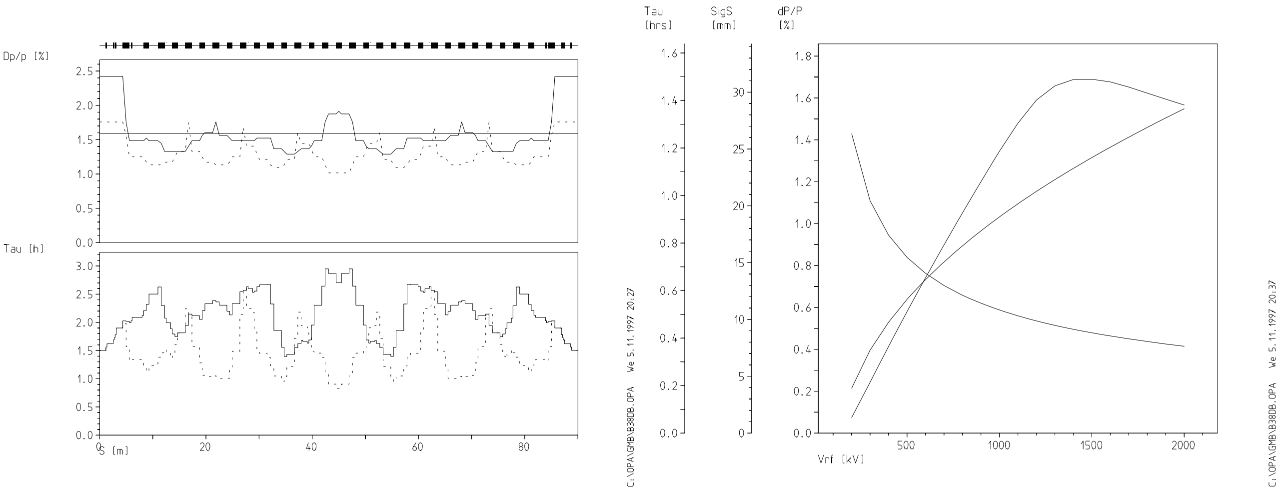
SLS (D1 mode, `T2.OPA`) : TRELMA = 5.27 %

SLS booster synchrotron



Beam optics of the SLS storage booster synchrotron

left: Betafunctions, horizontal (solid), vertical (dotted) and dispersion (top line) , one sixth of the lattice is shown.
right: Envelopes for 10% coupling



Momentum acceptance and Touschek lifetime results for the booster synchrotron

left: The upper plot shows the MA along one period of the lattice (one third of the ring), with the positive lattice MA (solid line), the negative lattice MA (dotted line) and the RF MA for 2.1 MV overvoltage (straight line), giving 4% of RF MA.
The lower plot shows the Touschek lifetime for each lattice element.

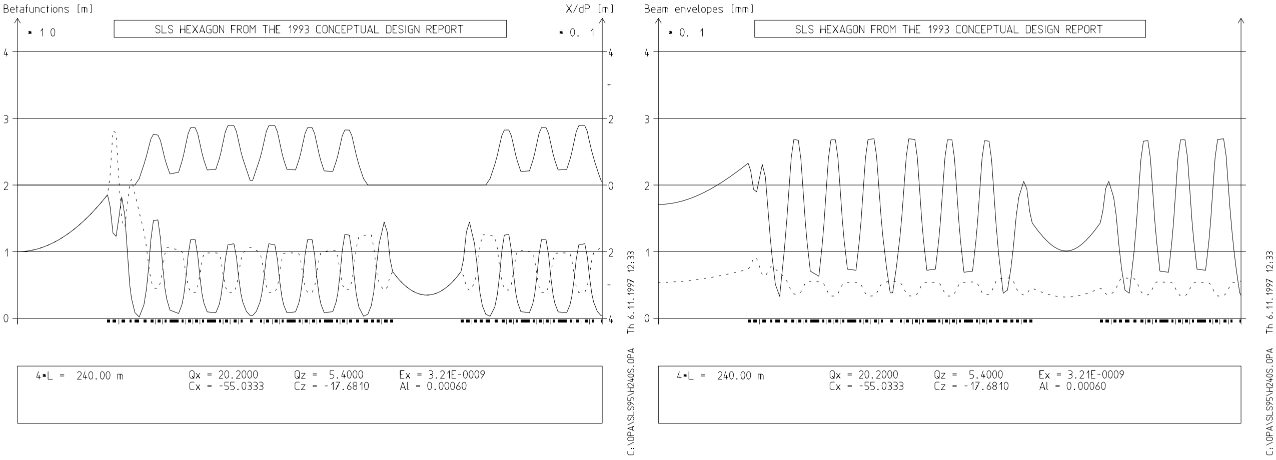
right: Touschek lifetime, RF momentum acceptance and bunchlength as a function of RF voltage.

Parameters: bunch charge q = 5 nCb (= 5.6 mA single bunch current), emittance coupling κ = 10 %
Beam pipe full width $2a_x$ = 30 mm for all elements (vertical is irrelevant here)

Comment: The lattice MA is limited by the narrow beampipe of 30 mm full width, the TRELMA from dynamic aperture limitation alone would be 1.61 %. Despite a rather large dynamic aperture the TRELMA is not larger, because the closed orbit has strong nonlinear distortions, leading to small off-momentum dynamic apertures.

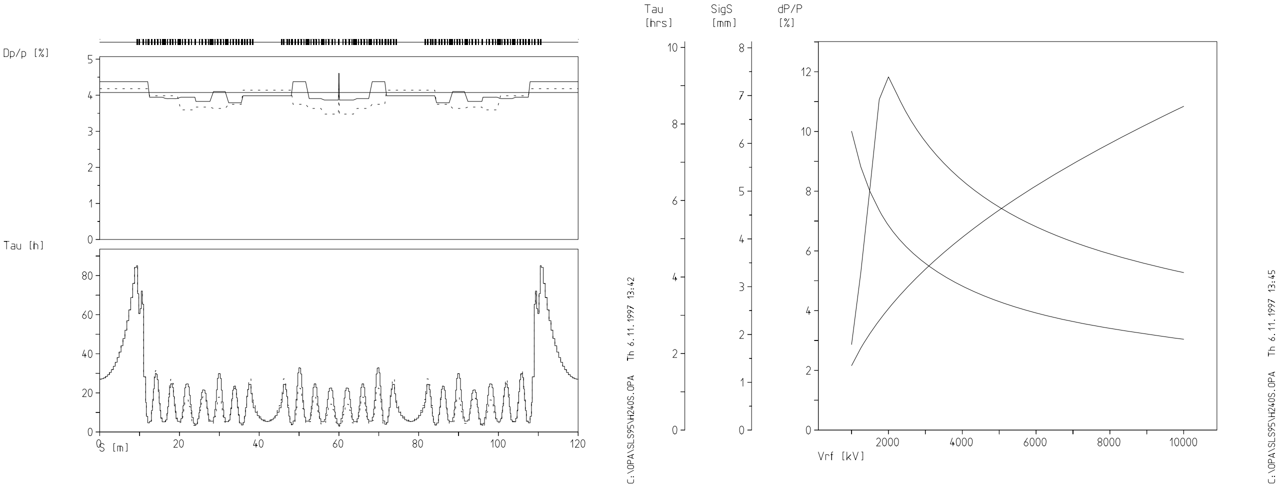
SLS Booster (`B38DB.OPA`) : TRELMA = 1.27 %

SLS hexagon lattice from 1993



Beam optics of the 1993 SLS hexagon

left: Betafunctions horizontal (solid), vertical (dotted), dispersion (top line) , one quarter of the lattice is shown.
right: Envelopes for 10% coupling



Momentum acceptance and Touschek lifetime results for the 1993 SLS hexagon

left: The upper plot shows the MA along one period of the lattice (half of the ring), with the positive lattice MA (solid line), the negative lattice MA (dotted line) and the RF MA for 2.0 MV overvoltage (straight line), giving 4% of RF MA.
The lower plot shows the Touschek lifetime for each lattice element.

right: Touschek lifetime, RF momentum acceptance and bunchlength as a function of RF voltage.

Parameters: Bunch charge q = 5 nCb (= 5.6 mA single bunch current), emittance coupling κ = 10 %
Beam pipe full width $2a_x$ = 70 mm for all elements (vertical is irrelevant here)

Comment: The MA is almost constant along the lattice and correspondingly the lifetime maximum is very sharp. The lattice MA is constant due to the fact, that the closed orbit ran into a stopband beyound ±4% momentum deviation, because it was impossible with that type of lattice to compensate the 2^{nd} order chromaticity – what eventually lead to its rejection [B96].

1993 SLS Hexagon (`H240S.OPA`) : TRELMA = 3.81 %

Touschek lifetime results

The following table diplays the expected Touschek lifetimes for SLS in different modes of operation, for the booster synchrotron and also for the old hexagon lattice:

SLS lattice mode	**E [GeV]**	**q [nCb]**	**κ [%]**	**V [MV]**	**τ [hrs]**
D0, standard (few bunch mode)	**2.1**	**5**	**10**	**2.1**	**7.94**
D0, more RF (add. s.c. cavities)	2.1	5	10	**4.2**	**12.7**
D0, lower current (multibunch mode)	2.1	**1**	10	2.1	**39.7**
D0, lower coupling	2.1	5	**1**	2.1	**2.68**
D0, higher energy	**2.4**	5	10	2.6	**13.6**
D1, standard (few bunch mode)	2.1	5	10	2.1	**7.37**
Booster synchrotron	2.1	5	10	0.5	**0.51**
SLS hexagon lattice from 1993	2.1	5	10	2.0	**9.23**

(with: **E** beam energy, **q** single bunch charge, **κ** emittance coupling, **V** RF voltage, **τ** Touschek lifetime)

For SLS/D0 at 2.1 GeV in few bunch mode, i.e. approx. 20 bunches, each wich a relatively high charge of 5 nCb, we expect approx. 8 hrs lifetime. Bunchlengthening was not included and would further increase the lifetime, the scaling being simply linear as long as the regime of momentum spread widening is not entered.

Doubling the RF voltage by installing passive superconducting cavities as proposed earlier [M96] would increase the RF MA to 6%, better matching to the TRELMA of 5.4% and increase the lifetime by 60% close to the maximum possible value of approx. 13 hrs.

An alternative to increase Touschek lifetime would be to run SLS on a higher energy of 2.4 GeV, while increasing the RF-voltage to the maximum capacity of the RF system to keep the RF MA at 4%. The resulting lifetime of 13.6 hrs indicates an energy scaling of Touschek lifetime of $\tau \propto E^4$, which was not so obvious from Eq.1.

However probably the cheapest alternative for increasing the Touschek lifetime would be to lengthen the bunch with [passive] Landau cavities operating at a higher harmonic, probably they will be required anyway to prevent coupled bunch instabilities.

In normal operation mode with approx. 400 mA full stored beam current distribute on 400 bunches of each approx. 1 nCb, the lifetime would be 40 hrs at 2.1 GeV.

The old hexagon design for SLS, though having much lower TRELMA and even lower emittance, shows comparable, even slightly better Touschek lifetime for the reason that it did not yet contain mini-beta straight, which are responsible for the highest particle losses.

Conclusions

- Full nonlinear calculation of the lattice momentum acceptance gives slightly lower Touschek lifetime results than ZAP, the differences amounting to approx. 5...10% for the relevant range of RF voltage.
- The Touschek relevant effective lattice momentum acceptance (TRELMA) was defined as a useful quantity to characterize the lattice performance in momentum acceptance by a single number. For SLS in standard mode (dispersionfree straight sections) is TRELMA = 5.4 %.
- Touschek lifetime for SLS at 2.1 GeV will be 8 hours in few bunch mode (5 nCb per bunch) and 40 hours in multibunch mode (1 nCb per bunch) for an emittance coupling of 10 %.
- Doubling the RF voltage by installing a superconducting cavity would increase the Touschek lifetime by 60%.
- Increasing the energy of SLS to 2.4 GeV (and keeping the RF acceptance constant) would increase the Touschek lifetime by 70%.
- Despite its narrow beampipe the booster synchrotron will still provide a Touschek lifetime of ½ hour for a 5 nCb bunch at 2.1 GeV.

Acknowledgements

I would like to thank Amor Nadji and Phi Nghiem at SOLEIL for very helpful discussions.

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