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Non-Linearities in Light Sources

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Correction of a light source's large chromaticity while maintaining sufficient dynamic aperture and momentum acceptance requires careful optimization of the sextupole configuration in first and second order of sextupole strength.
In this paper we try to explain chromaticity and its correction in a most intuitive way. Accompanied by the step wise improvement of a test lattice, the scheme is then expanded from plain correction with two sextupole families to installation of "harmonic" setupole for first and second order optimization. The approach is most visual and pragmatic on expense of the mathematical formalism which may be found in the references.

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1 Introduction

Sextupole magnets for correcton of chromaticities are the dominating nonlinear elements in a light source lattice, because light sources are built for lowest emittance and thus require very strong horizontal focussing. Strong quadrupoles cause large chromaticities which require strong sextupoles for compensation in order to avoid single bunch head tail instability and several multi bunch instabilities.

The parabolic field variation in a sextupole makes it an essentially nonlinear device which causes chaotic or unbounded particle motion beyond some maximum stable amplitude. The phase space area enclosing stable particle oscillations is called dynamic acceptance, its projection onto physical space dynamic aperture.

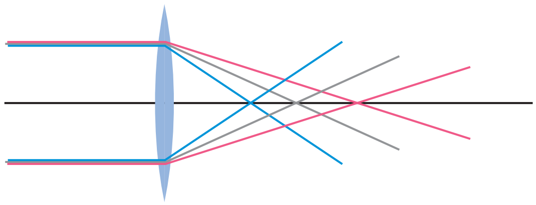
Light sources require a large horizontal dynamic aperture for injection. In particular top-up operation depends on clean and efficient injection. Light sources also require a large momentum acceptance for sufficient beam life time: The Touschek scattering contribution to beam life time is usually large in light sources because due to small beam emittance the particle density in the bunch is high raising the probability for intra-beam scattering events with large momentum changes. Vertical dynamic aperture is less relevant, because the physical apertures will be rather low anyway due to the presence of low gap insertion devices. Obviously this also prohibits any scheme for vertical injection. Typical requirements are about $10 \ldots 30$ mm·mrad for horizontal acceptance, about $1 \ldots 5$ mm·mrad for vertical acceptance and about $\pm 3 \ldots 4$ % for relative momentum acceptance.

Thus light source design has to face the conflict between a requirement for large dynamic acceptance and the unavoidable presence of strong nonlinear devices attacking exactly this acceptance. It is no exaggeration to consider this problem the most challenging task in light source lattice design.

Apart from the sextupoles, other nonlinearities have to be taken into account, like higher multipoles in ring magnets due to saturation or geometric imperfections, and in insertion devices due to finite pole width and other imperfections. But these nonlinearities can be suppressed by proper design of components, and tolerance limits will be provided based on tracking calculations. But the sextupoles, to make this clear, are nonliner *by design* and require appropriate treatment. This is subject of this paper. We will proceed in four steps: Sec. 2 will explain how quadrupoles generate chromaticity, sec. 3 will show, how chromaticity is corrected with two families of sextupoles, and how the dynamic acceptance is destroyed. Sec. 4 will explain how introducing several more sextupole families helps to suppress the adverse effects while maintaining chromaticity corrections and how to do it. Sec. 5 considers second order sextupole effects which are also important for low emittance light sources and how further dynamic acceptance optimization can be achieved.

2 Chromaticity

As sketched here, a quadrupole shows chromatic aberrations, i.e. a variation of focal length with momentum. Since its strength is given by



$$b_2 = \frac{1}{(B\,\rho)}\,\frac{d\,B_y}{d\,x}$$

with the "magnetic rigidity"

$$(B\,\rho) := \frac{p}{e} = 3.3356\,\mathrm{Tm}\,\cdot\,E[\mathrm{GeV}]$$

a function of momentum, the focussing strength varies as

$$b_2(\delta) = \frac{b_2}{(1+\delta)} \approx b_2\,(1-\delta) \qquad\qquad \delta := \frac{\Delta p}{p} \ll 1$$

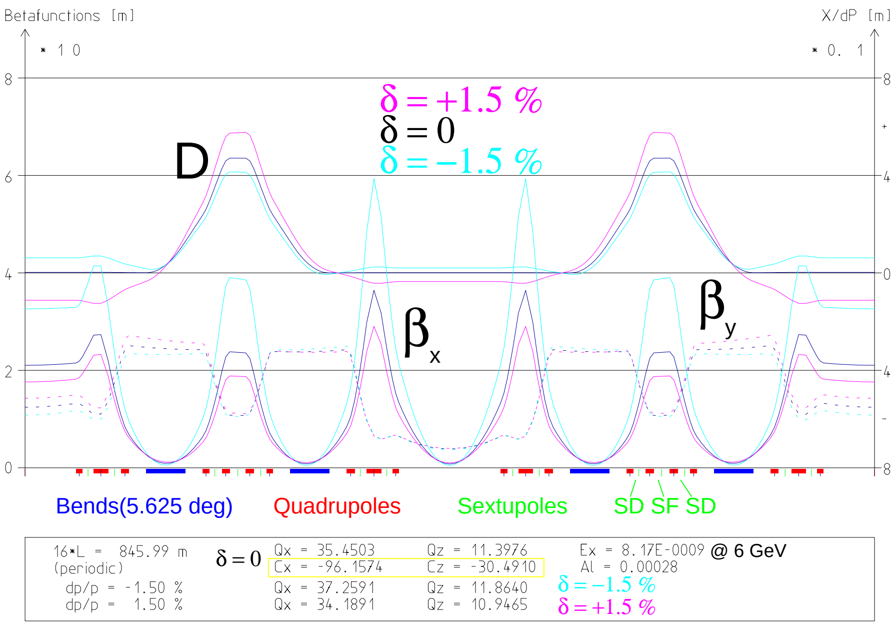


Figure 1: A double-cell (two double bend achromats out of 32) of the original ESRF lattice with dispersionfree straight sections as a typical test case for light source lattices. Uncorrected chromaticity results in strong variation of beta functions with momentum.

As a consequence, the off-momentum optical functions in a strong focusing lattice as shown in fig.1 vary significantly from the on-momentum functions even for small momentum deviations. Chromaticity is defined as the variation of tune with momentum and derived in the most simple way as a gradient distortion to the one turn matrix [6]:

$$\begin{pmatrix} 1 & 0 \\ \pm b_2 \delta \, ds & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\frac{\sin 2\pi Q}{\beta} & \cos 2\pi Q \end{pmatrix} = \begin{pmatrix} \cos 2\pi \tilde{Q} & \beta \sin 2\pi \tilde{Q} \\ -\frac{\sin 2\pi \tilde{Q}}{\beta} & \cos 2\pi \tilde{Q} \end{pmatrix} := \tilde{\mathcal{M}}$$

The new tune is given by the half trace of the new matrix:

$$\frac{1}{2}\mathrm{Tr}(\tilde{\mathcal{M}}) = \cos 2\pi \tilde{Q} = \cos 2\pi (Q + \Delta Q) = \cos 2\pi Q \pm \frac{1}{2} b_2 \delta \, \beta \, \sin 2\pi Q \, ds$$

Considering small deviations in tune, the cosine is expanded and we obtain the horizontal and vertical chromaticity $\xi_{x/y}$ by integration over all gradient distortions over the ring:

$$\Delta Q \ll 1 \longrightarrow \Delta Q = \mp \frac{1}{4\pi} b_2 \delta \, \beta \, ds \qquad \xi_{x/y} = \frac{\Delta Q}{\delta} = \mp \frac{1}{4\pi} \oint_C b_2(s) \, \beta_{x/y}(s) \, ds$$

Both chromaticities are naturally negative, where in a light source in particular the horizontal chromaticity will become large in absolute value: values can be in the order of $\xi_x \approx -100$. Negative chromaticity must be avoided for suppression of the head-tail instability, an instable oscillation between leading and trailing electrons inside a bunch with onset at very low currents [5]. Also coupled bunch oscillations are driven by negative chromaticity. On the other hand, a large absolute value of (negative or positive) chromaticity, would result in a wide tune spread of the beam halo leading to particle losses at low order resonances and thus low momentum acceptance. For that reason, the chromaticities have to be zero or at moderate positive values.

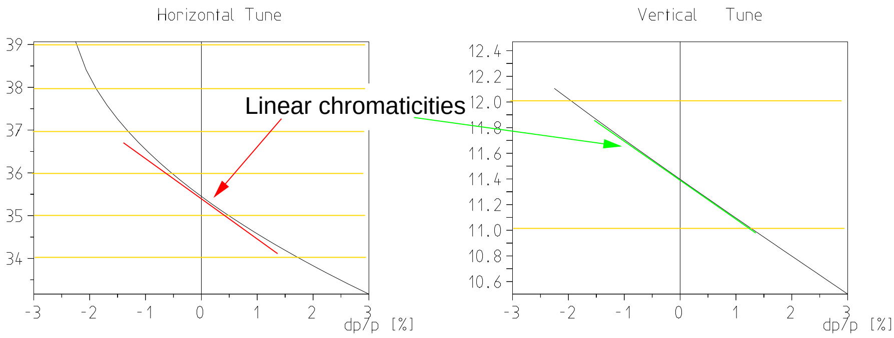
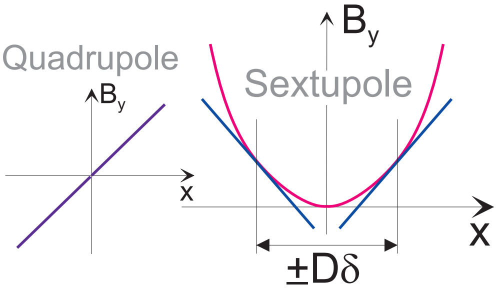


Figure 2: Variation of betafunctions and tunes with momentum without chromaticity correction for the lattice shown in fig.1. Note how the large horizontal chromaticity leads to a wide tune spread extending over several integers!

Fig.2 shows the large variation of tunes with momentum due to natural chromaticity. Considering that particles get lost at least at integer and half integer resonances, the momentum acceptance for the example shown would amount to less than 0.5 % even for very low currents. Reasonable currents couldn't be stored anyway due to head-tail instability.

3 Sextupoles for chromaticity correction



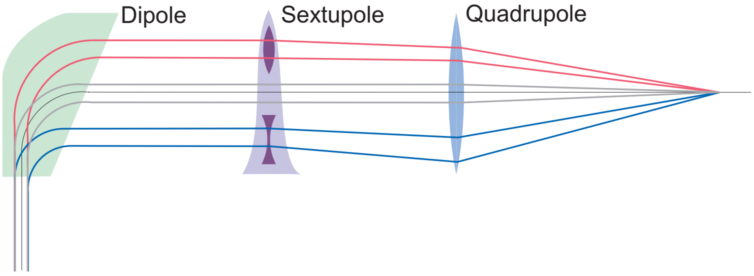
The magnetic field in a sextupole varies as

$$B_y\,(x) = \frac{1}{2} B''\,x^2$$

Thus in a small range around some $\hat{x}$ it can be considered like a quadrupole with gradient

$$B_y'(\hat{x})\ = B'' \cdot \hat{x}$$

and thus adds or subtracts focussing strength depending on $\hat{x}$. Dispersion D as generated by a dipole magnet sorts particles by momentum:



Thus, if

$$\hat{x} = D\delta$$

and the sextupole strength well chosen, the chromatic aberration of a quadrupole can be compensated:

The kick on a particle in a quadrupole, resp. sextupole is given by

$$\begin{array}{lrcl} \text{Quadrupole:} & \Delta x' & = & -b_2 L x \\ & \Delta y' & = & b_2 L y \\ \text{with} & b_2 & = & \frac{1}{(B\,\rho)}\frac{d\,B_y}{d\,x} \end{array} \qquad \begin{array}{lrcl} \text{Sextupole:} & \Delta x' & = & -b_3 L (x^2 - y^2) \\ & \Delta y' & = & 2 b_3 L x y \\ \text{with} & b_3 & = & \frac{1}{2}\frac{1}{(B\,\rho)}\frac{d^2\,B_y}{d\,x^2} \end{array}$$

Chromatic aberrations modify the magnet strength as

$$b_n(\delta) = b_n/(1+\delta) \approx b_n(1-\delta)$$

Adding dispersion to the beam is a transformation

$$x \longrightarrow D\delta + x \qquad y \longrightarrow y$$

Inserting these transformed coordinates into the kick equations and keeping up to second orders in products of x, y and δ gives

$$\begin{array}{lrcl} \text{Quadrupole:} & \Delta x' & = & -b_2 L x + [b_2 L]\,\delta\,x \qquad \Delta y' = +b_2 L y - [b_2 L]\,\delta\,y \\ \text{Sextupole:} & \Delta x' & = & -[2 b_3 L D]\,\delta\,x - b_3 L (x^2 - y^2) - b_3 L D^2 \delta^2 \\ & \Delta y' & = & +[2 b_3 L D]\,\delta\,y + 2 b_3 L\,x y \end{array}$$

Obviously for $2b_3 LD \stackrel{!}{=} b_2 L$ the sextupole corrects the quadrupole's chromaticity. However there are other terms coming with the sextupoles which cause the problems with dynamic apertures.

This simple example has to be generalized to quadrupolar and sextupolar fields distributed around the ring. Chromaticities are then given as integral expressions:

$$\xi_{x/y} = \pm\frac{1}{4\pi}\oint_C [2b_3(s)D(s) - b_2(s)]\,\beta_{x/y}(s)\,ds \tag{1}$$

Considering discrete, thin magnets, i.e. neglecting any change of optical functions over the magnet length, this equation can be expressed as a sum:

$$\xi_{x/y} = \frac{1}{4\pi}\left(\pm\sum_{\text{sext}} 2(b_3L)_n\,\beta_{(x/y)n}\,D_n \mp \sum_{\text{quad}} (b_2L)_n\,\beta_{(x/y)n}\right) \tag{2}$$

Defining **2** *families* of sextupoles **SF, SD** for correction of horizontal and vertical chromaticity, this equation can be written as a 2×2 linear system for the 2-vector of sextupole strengthes:

$$\frac{1}{2\pi}\begin{pmatrix} +\sum_{n\in\mathrm{SF}}\beta_{xn}D_n & +\sum_{n\in\mathrm{SD}}\beta_{xn}D_n \\ -\sum_{n\in\mathrm{SF}}\beta_{yn}D_n & -\sum_{n\in\mathrm{SD}}\beta_{yn}D_n \end{pmatrix}_{2\times 2} \cdot \begin{pmatrix} (b_3L)_{\mathrm{SF}} \\ (b_3L)_{\mathrm{SD}} \end{pmatrix}_{1\times 2}$$

$$= \frac{1}{4\pi}\begin{pmatrix} +\sum_{\mathrm{Quad}}(b_2L)\,\beta_{xn} \\ -\sum_{\mathrm{Quad}}(b_2L)\,\beta_{yn} \end{pmatrix}_{1\times 2} \begin{bmatrix} + & \Delta\xi_x \\ & \Delta\xi_y \end{bmatrix} \Longrightarrow \quad (b_3L)_{\mathrm{SF}},\,(b_3L)_{\mathrm{SD}} \tag{3}$$

Here $[\Delta\xi_{x/y}]$ indicates an optional non-zero value of chromaticity to be obtained from the sextupoles. From the linear system a first recommendation how to place the sextupoles becomes visible: All sextupoles should be at locations of large dispersion. In order to decouple the two families, i.e. to avoid that they act against each other, or, mathematically speaking, that the linear system degenerates, the SF-family members should be located at locations of large β_x and low β_y, and the SD-family members at locations of large β_y and low β_x. Fig. 1 shows the placement of the sextupoles: Usually finding a good place for SF is easy, since $\beta_x = x^2/\epsilon$ and dispersion $D = x/\delta$ are both subject to horizontal focussing and thus behave similar: In fig. 1 the central DBA quadrupole required for reflecting the dispersion function was split into two quadrupoles and the SF-sextupole inserted between. For SD-sextupoles locations with optimum conditions are hard to find: In fig. 1 two SDs are inserted at medium dispersion and β_y slightly larger than β_x.

Fig. 3 shows the result of successful chromaticity correction. But tracking particles in phase space reveals how the dynamic aperture breaks down as shown in the Poincaré plots of fig.4: The horizontal dynamic aperture is too small to inject a beam into the machine, also the coupling from vertical to horizontal is significant.

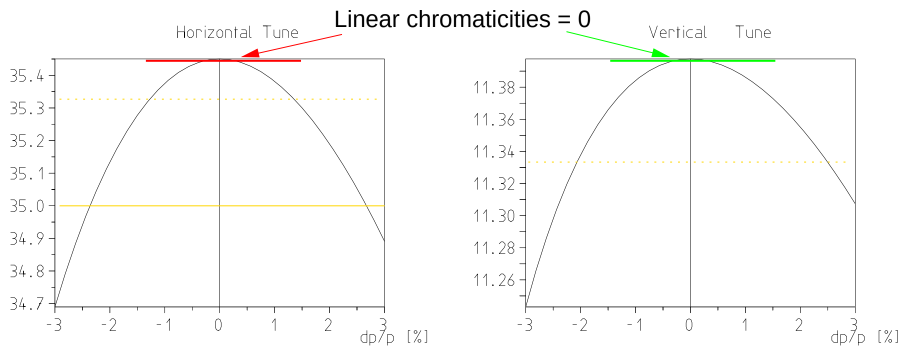


Figure 3: Variation of betafunctions and tunes with momentum after chromaticity correction with 2 families of sextupoles: Linear chromaticity has disappeared as visible by the horizontal tangent to the tune vs. dp/p curves, but second order chromaticity remains.

4 First order sextupole optimization

The lattice with plain chromaticity emerging from the previous section would not be able to operate, because the dynamic aperture is too small to inject a beam, and the beam lifetime would be very short. But chromaticity correction is mandatory in order to store significant current in the machine. So, the method of chromaticity correction has to be improved in order to suppress the adverse sextupole effects.

In the following we will outline a "standard procedure" of sextupole optimization which is derived and described in detail in refs. [2, 3, 4]:

In order to understand, what sextupole actually do to the beam and how they destroy it, and in order to find a cure subsequently, the single particle Hamiltonian has to be studied. In a lattice made from dipoles, quadrupoles and sextupoles it is given by

$$H(s) = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{\text{kinetic}} \overbrace{- \underbrace{b_1\, x\, \delta}_{\text{dispersive}} + \underbrace{\frac{b_1^2}{2}\, x^2}_{\text{focussing}}}^{\text{Dipoles}} + \underbrace{\overbrace{\frac{b_2}{2}\,(x^2 - y^2)}^{\text{Quadrupoles}}}_{H_2(s)} + \underbrace{\overbrace{\frac{b_3}{3}\,(x^3 - 3xy^2)}^{\text{Sextupoles}}}_{H_3(s)} \tag{4}$$

The goal is finding a sextupole distribution such that $H_2 + H_3$ becomes achromatic while staying linear. For that purpose, the linear betatron oscillation for a flat (i.e. $D = D_x$, $D_y = 0$) lattice, given by

$$x(s) = \sqrt{2J_x\, \beta_x(s)} \cos\phi(s) + D(s)\,\delta \qquad y(s) = \sqrt{2J_x\, \beta_x(s)} \cos\phi(s) \tag{5}$$

is introduced into eq.4, the powers of the trigonometric functions are turned into linear functions of multiple arguments, and the different modes, i.e. terms with same arguments are collected. As a result, the quadrupole/sextupole Hamiltonian can be represented by a sum over different frequencies:

$$\int_{\text{cell}} [H_2(s) + H_3(s)]\, ds = \sum h_{jklmp}, \quad \text{with}$$

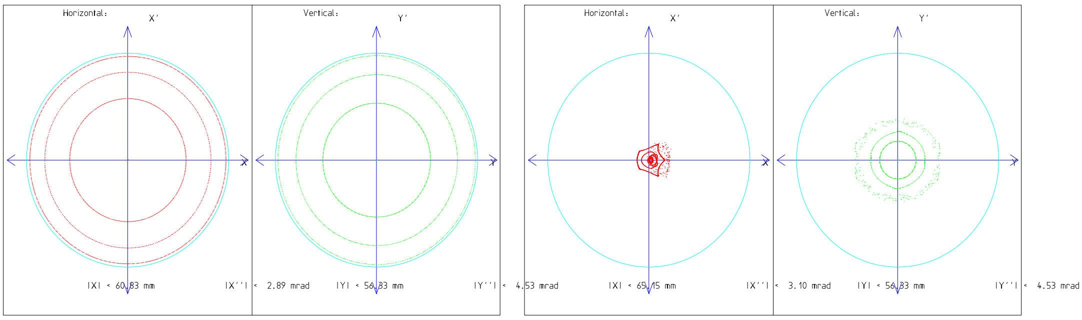
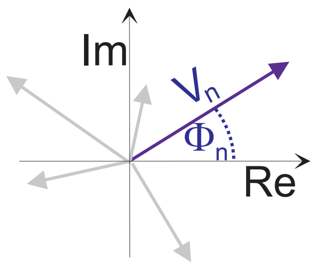


Figure 4: Poincaré plots of particle motion in horizontal (red) and vertical (green) phase space before (left) and after (right) plain chromaticity correction with 2 families of sextupoles: Without sextupoles the motion is linear and the dynamic acceptances are unlimited. With sextupoles, nonlinear effects lead to a breakdown of dynamic acceptances. The bright blue circles indicate the physical aperture from an unrealistic quadratic beampipe of 160 mm width. (Real beampipes are typically $60\ldots90$ mm wide and $20\ldots40$ mm high.)

$$h_{jklmp} \;\propto\; \begin{array}{l} \sum_n^{N_{\text{sext}}} (b_3\,L)_n\,\beta_{xn}^{\frac{j+k}{2}}\;\beta_{yn}^{\frac{l+m}{2}}\;D_n^p\;e^{i\{(j-k)\phi_{xn}+(l-m)\phi_{yn}\}} \\ -\left[\sum_n^{N_{\text{quad}}} (b_2\,L)_n\,\beta_{xn}^{\frac{j+k}{2}}\;\beta_{yn}^{\frac{l+m}{2}}\;e^{i\{(j-k)\phi_{xn}+(l-m)\phi_{yn}\}}\right]_{p\neq 0} \end{array} \tag{6}$$

This complex expression becomes more intuitive when abbreviating

$$h = \sum_n^{N_{\text{sext}}} V_n e^{i\Phi_n}\;[\,+\ldots\text{quads for } p\neq 0\ldots]$$



The Hamiltonian modes are sums of complex vectors, where each vector corresponds to a sextupole, its length given essentially by the sextupole's integrated strength (multiplied with the optical functions at its location which is the "lever arm" to act on the beam), and its complex phase by the [modal multiple] of the betatron phases at its location.

All together, nine Hamiltonian modes are found: Two of them have zero betatron phases, They are given by

$$\begin{aligned} h_{11001} &= +J_x\delta\,[\,\sum_n^{N_{sext}} (2b_3\,L)_n\beta_{xn}D_n - \sum_n^{N_{quad}} (b_2\,L)_n\beta_{xn}\,] \\ h_{00111} &= -J_y\delta\,[\,\sum_n^{N_{sext}} (2b_3\,L)_n\beta_{yn}D_n - \sum_n^{N_{quad}} (b_2\,L)_n\beta_{yn}\,] \end{aligned}$$

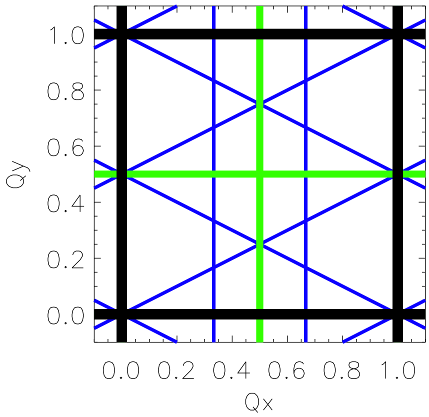
Comparison to eq.2 reveals, that these are just the chromaticities!

Seven modes (and their complex conjugates) have phase arguments, thus they do not add up over many turns but they will show a *resonant* structure: Long term behaviour is revealed by considering N repetitions of the lattice structure, i.e. many cells and many turns, and extrapolating to infinity:

$$\begin{aligned} |h^{\infty}_{jklmp}| &= \left| \sum_{n=0}^{\infty} h_{jklmp} e^{i\,\{(j-k)\,2\pi\,n\,Q_x^{\text{cell}} + (l-m)\,2\pi\,n\,Q_y^{\text{cell}}\}} \right| \\ &= \frac{|h_{jklmp}|}{2\sin\pi[(j-k)\,Q_x^{\text{cell}} + (l-m)\,Q_y^{\text{cell}}]} \end{aligned} \tag{7}$$

In particular we find modes driving the following resonances:

$$\begin{aligned} h_{21000} &= h^*_{12000} \longrightarrow \mathbf{Q_x} \\ h_{30000} &= h^*_{03000} \longrightarrow 3\,Q_x \\ h_{10110} &= h^*_{01110} \longrightarrow \mathbf{Q_x} \\ h_{10200} &= h^*_{01020} \longrightarrow Q_x + 2\,Q_y \\ h_{10020} &= h^*_{01200} \longrightarrow Q_x - 2\,Q_y \\ h_{20001} &= h^*_{02001} \longrightarrow 2\,Q_x \\ h_{00201} &= h^*_{00021} \longrightarrow 2\,Q_y \end{aligned}$$



These resonances correspond to forbidden lines in the tune diagram, i.e. working points close to these lines must be avoided, otherwise particles will be resonantly extracted from the beam and get lost. The last two are chromatic half integer resonance drive terms and cause chromatic variation of the beta functions.

As long as the periodicity of the machine is large and non-linearities are not too strong, there may be one or a few single resonances dominating the dynamics. In this case, the calculation may be continued further by harmonic expansion of the hamiltonian modes in order to obtain single resonance drive terms [1]. The traditional approach to suppress a single or a small group of terms (while exciting others) by appropriate distribution of sextupoles had been proven successful in the past for several machines. However, a very advanced light source may have rather low periodicity, thus the tune space is densely covered with sextupolar and other resonances, and many resonances will contribute to the dynamics. In this more general case, it is better to suppress the h_{jklmp}-modes right from the beginning because they are the source of *all* the resonances.

This is done by adding more sextupoles, even in dispersion-free regions, which do not contribute to chromaticity correction, but are solely installed for mimimization of the resonance driving modes. Four families of these – traditionally called – “harmonic” sextupoles are visible in the dispersion free matching sections in fig.1. Since each sextupole corresponds to a complex vector, diagrams like shown in fig. 5 can help to suppress the resonance driving modes while keeping the chromaticities compensated.

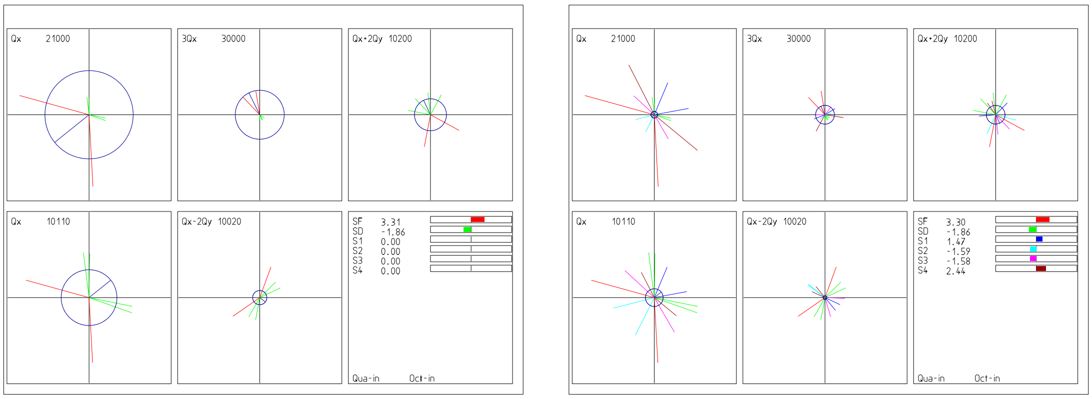
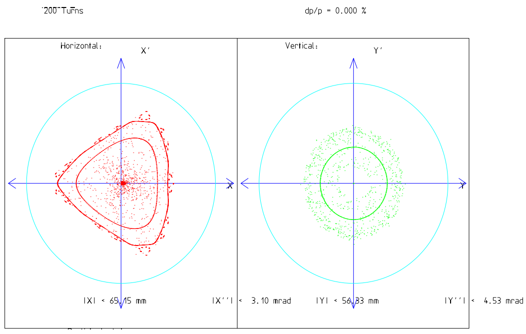
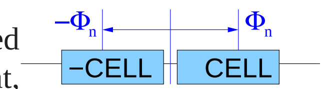


Figure 5: Complex vector diagrams for the five first order non-chromatic sextupole Hamiltonian modes. Each vector corresponds to a sextupole. The dark blue vectors and circles mark the resulting vector sums h_{jklm0}. Left figure shows the effect of plain chromaticity correction, right figure the situation after adding four families of harmonic sextupoles.



Even after this simple manual optimization using four additional sextupole families, the gain in dynamic acceptance is impressive when comparing the result of phase space tracking with fig. 4 (right), which was obtained without harmonic sextupoles.
Note, that the harmonic sextupoles are no small correctors but of comparable strength to the chromaticity sextupoles!
Another improvement is visible in the frequency spectra of a test particle before and after installing harmonic sextupoles as shown in fig. 6: Amplitudes of peaks related to sextupolar resonances are significantly reduced in height.

This procedure can be set up more systematic: There are 9 modes to be adjusted, 2 of them real, 7 complex, which gives a total of 16 quantities to be minimized.



However, most machines have symmetry points usually used as reference for tracking etc. Seen from such a symmetry point, every element has a mirror image of opposite betatron phase.

Thus the complex parts cancel, and we are left with only 9 equations, which in principle could be solved by means of 9 sextupole families.

These equations ($\to$ eqs.6) are linear in sextupole strength. If there are M sextupole families, we get a $9 \times M$ linear system for the vector of sextupole strengths:

$$\left\{ \sum_{n \in \{Sm\}} \beta_n^{(\ldots)} D_n^{(\ldots)} e^{i\{(\ldots)\phi_n\}} \ldots \right\}_{9\times M} \cdot \left\{ (b_3 L)_m \right\}_{M\times 1} = \left\{ \sum_{\text{Quad}} (b_2 L) \ldots \right\}_{1\times 9} \tag{8}$$

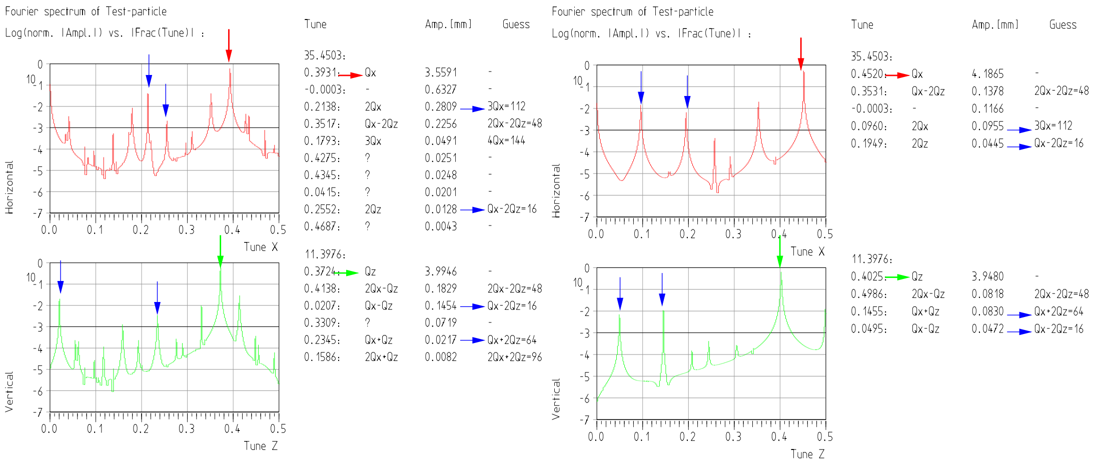


Figure 6: Fourier spectra of a test particle starting at $x_o = y_o = 4$ mm, $x'_o = y'_o = \delta = 0$ and tracked over 512 turns with harmonic sextupoles switched off (left) and on (right). Note how the amplitude of peaks related to sextupolar resonances decrease. Also higher order effects, like shift of fundamental tune and higher order resonances are reduced significantly.

This is just a generalization of the 2×2 system for plain chromaticity correction from eq. 3. For $M = 9$ the system is quadratic and can be solved exactly, for $M \neq 9$ the method of singular value decomposition (SVD) [9] is able to obtain a solution too, which for $M < 9$ returns a least square minimized approximation to the right hand side vector, resp. for $M > 9$ selects the solution associated with minimum sextupole strength. SVD also provides insight into the good behaviour of the linear system by returning a vector of weighting factors. Degeneration of the system is indicated by the appearance of very small weighting factors.

In fact, in particular for light sources, the system tends to degenerate, as investigated in detail in ref.[3]: The horizontal betatron phase advance per cell in a typical light source lattice is $\Delta\phi_x^{\text{cell}} \approx 140\ldots160°$ in order to obtain low emittance ($\rightarrow$ sec.4.4 and fig.7 in [11]). This is quite close to 180°, thus $e^{i2\phi_x} \approx 1$. As a consequence the $2Q_x$ resonances driving term h_{20001} becomes proportional to the chromaticity $\xi_x \propto h_{11001}$. This means: no sextupole pattern exists to suppress this term *and* ξ_x.

$2Q_x$ resonances cause a momentum dependant beta-beat $\partial\beta_x/\partial\delta$, which is source of second order chromaticity $\partial^2 Q_x/\partial\delta^2$. In fact, light source lattices are often limited in momentum acceptance due to large second order chromaticity driving off-momentum particles to resonances. Even after installation of harmonic sextupoles, second order chromaticities remain large as visible by the parabolic shape of the tune vs. momentum curves:

In order to avoid or cure this degeneration problem, either particular betatron phase relations between sextupoles have to be fixed in order to exploit periodicity and symmetry conditions, or a distributed dispersion optics has to be chosen in order to make

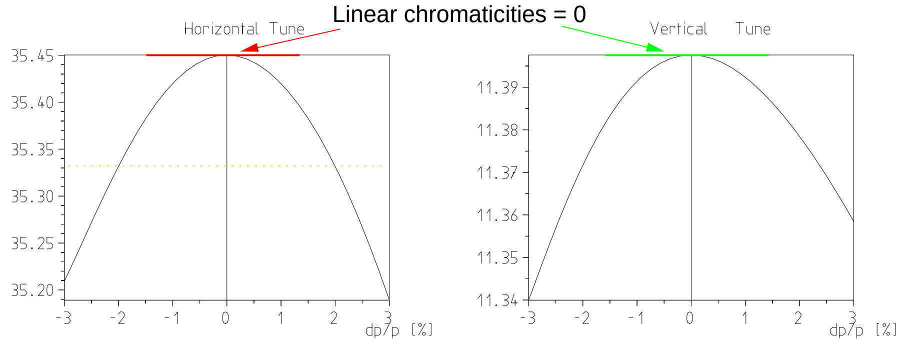
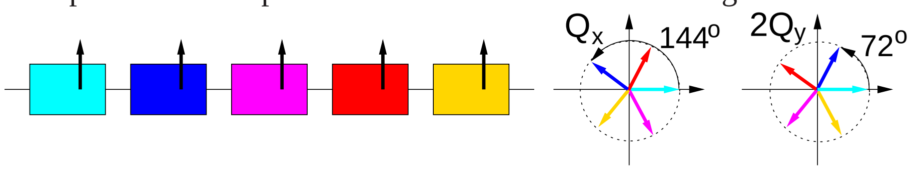


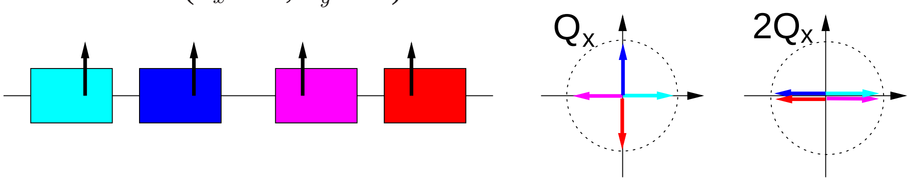
Figure 7: Dynamic acceptances after first order sextupole opimization. The dotted yellow line indicates the $3Q_x = 106$ resonance.

all sextupoles chromatic. In any case, these measures have strong impact on linear lattice design and layout:

Periodicity: If there are N cells, the ideal tune advances per cell would simultaneously make $N\Delta Q_x^{\rm cell}$, $3N\Delta Q_x^{\rm cell}$, $2N\Delta Q_x^{\rm cell}$, $2N\Delta Q_y^{\rm cell}$ integers. As an example, this works with $N = 5$, $\Delta Q_x^{\rm cell} = 0.4\,(= 144^\circ)$, $\Delta Q_y^{\rm cell} = 0.1\,(= 36^\circ)$, where 144° phase advance per cell is a reasonable value for a light source lattice:



Symmetry: A lattice section vs. its mirror image may suppress $2Q_x$, $2Q_y$ if the tune advances per section are close to $\Delta Q_x^{\rm cell} = \frac{2n_x+1}{4}$, $\Delta Q_y^{\rm cell} = \frac{2n_y+1}{4}$ (n_x, n_y integers). Two sections vs. their mirror also suppress Q_x, $3Q_x$. Of course this requires fixed phase advances over the straight sections which restricts the lattice flexibility. As an example the scheme applied to the Swiss Light Source [4] is sketched here ($n_x \approx 7$, $n_y \approx 3$) :



Dispersive straights: If dispersion in the straight sections is accepted, also the harmonic sextupoles become chromatic sextupoles. Thus several families add up in chromaticity correction but may be tuned to cancel their contribution to the $2Q_x$ terms, i.e. the linear system from eq.8 is *not* degenerate! This is an advantage of distributed dispersion lattices like SOLEIL [8], which in fact achieved a very good suppression of second order chromaticity.

Single resonance suppression: If, in the end, a single resonance is responsible for limitation of momentum acceptance, a harmonic decomposition of the source mode may be performed for suppression of this particular resonance. This was successfully done for the $3Q_x = 106$ resonance crossed at $\delta \approx \pm 2\%$ in the original ESRF lattice (see figure on previous page) [10].

5 Second order sextupole optimization

Second order sextupole effects are due to crosstalk between different sextupoles: First order effects perturb the linear betatron motion from eq.5. Introducing the perturbed motion again into the sextupole Hamiltonian will lead to effects of second order in sextupole strength - and so on to even higher orders. Basically sextupoles can drive resonances of any order, but the drive terms become weaker with increasing order.

In this context we should mention, that *thick* sextupoles, i.e. where phases and betas vary over the length of the sextupole talk to themselves, i.e. there is crosstalk between slices of the same magnet. But here we consider only thin sextupoles.

Derivation of the (quite unwieldy) formulae is either done by second order perturbation theory [2] or by application of Lie algebra techniques [3]. Basically the second order modes are products of the first order modes from eq. 6 of type $h_{jklmp} \cdot h_{j'k'l'm'p'}$

It turns out, that there are five phase independant terms, two of them are the second order chromaticites which already had appeared in the previous section, and three are the amplitude dependant tune shifts:

$$\xi_x^{(2)} = \frac{\partial^2 Q_x}{\partial \delta^2} \quad \xi_y^{(2)} = \frac{\partial^2 Q_y}{\partial \delta^2} \qquad \frac{\partial Q_x}{\partial J_x} \quad \frac{\partial Q_x}{\partial J_y} = \frac{\partial Q_y}{\partial J_x} \quad \frac{\partial Q_y}{\partial J_y}$$

Second order chromaticites limit momentum acceptance. The analytic formulae based on chromatic beta-functions and higher order dispersion are rather lengthy, and codes for minimizing them run more efficient by obtaining the values from numerical differentiation of the dispersive closed orbit [3].

Amplitude dependant tune shifts lead to a twist in phase space and subsequent resonance overlap, and with it chaotic motion and fractal dynamic acceptance structure [7]. But a closed expression suitable for minimization is available (eq.(119) in [3] or eq.(195) in [2]).

There are also eight phase dependant terms. Calculating the long term behaviour analogous to eq.7 reveals resonance denominators of type

$$|h^{\infty}_{jklmpj'k'l'm'p'}| \propto \frac{(\ldots)}{8 \sin \pi[\vec{m} \cdot \vec{Q}] \sin \pi[\vec{m}' \cdot \vec{Q}] \sin \pi[(\vec{m} + \vec{m}') \cdot \vec{Q}]} \quad \text{with}$$

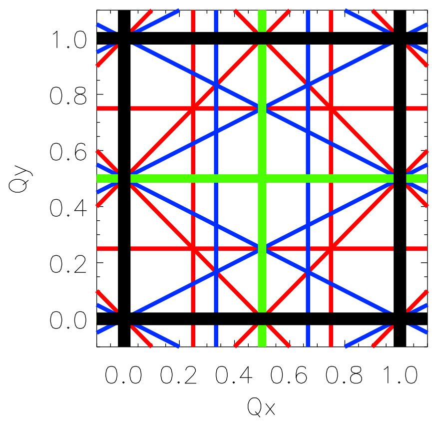
$$\vec{m} := \begin{pmatrix} j-k \\ l-m \end{pmatrix} \qquad \vec{Q} := \begin{pmatrix} Q_x^{\text{cell}} \\ Q_y^{\text{cell}} \end{pmatrix}$$

The third term in the denominator drives octupolar resonances, where the corresponding mode of the octupole Hamiltonian (i.e. the *first* order Hamiltonian H_4 of octupole magnets!), $h_{\hat{j}\hat{k}\hat{l}\hat{m}0}$ is identified by

$$
\begin{aligned}
\hat{j} &= (j-k)+(j'-k') \\
\hat{k} &= (j-k)-(j'-k') \\
\hat{l} &= (l-m)+(l'-m') \\
\hat{m} &= (l-m)-(l'-m')
\end{aligned}
$$

Following resonances are found:

$$
\begin{aligned}
h_{40000} &\rightarrow 4Q_x & \qquad h_{31000} &\rightarrow 2\,Q_x \\
h_{00400} &\rightarrow 4\,Q_y & \qquad h_{20110} &\rightarrow 2\,Q_x \\
h_{20200} &\rightarrow 2Q_x + 2\,Q_y & \qquad h_{00310} &\rightarrow 2\,Q_y \\
h_{20020} &\rightarrow 2Q_x - 2\,Q_y & \qquad h_{01110} &\rightarrow 2\,Q_y
\end{aligned}
$$



Practically, optimizing the dynamic aperture in a light source lattice requires a code for minimization of first and second order sextupole terms, and many iterations with well chosen weighting factors. Usually, the first order modes and the amplitude dependant tune shifts require large weights, whereas the octupolar resonances seem less important. The second order chromaticities turn out to be quite resistive against optimization, because their source are inappropriate phase advances between lattice sections. Of course, the chromaticities are kept constant during minimization by executing eq.3 after each step.

In fact, second order optimization helps to improve the first order result as proven by increased dynamic aperture in fig. 8 (left), and further reduced tune walk with momentum as to be seen from fig. 9 in comparison to fig. 7.

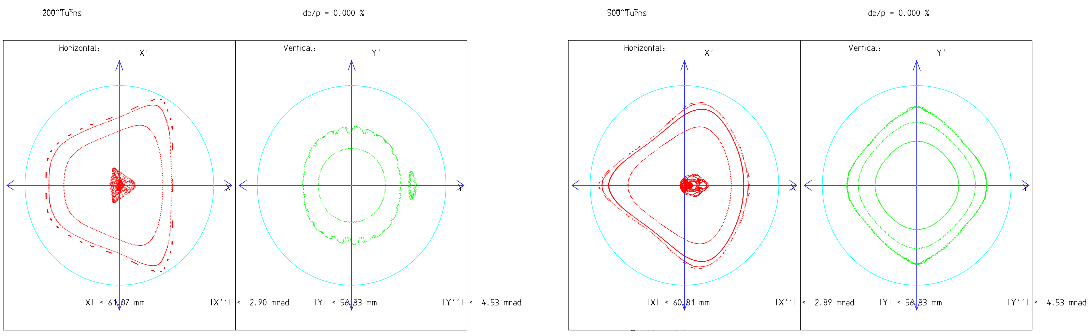


Figure 8: Dynamic acceptance improvement by second order optimization of 4 harmonic sextupole families (left), and further improvement after adding small octupoles (right)

However, basically it appears unnatural and inefficient to optimize the sextupoles while putting most weight on their *second* order effects. Instead octupoles could be

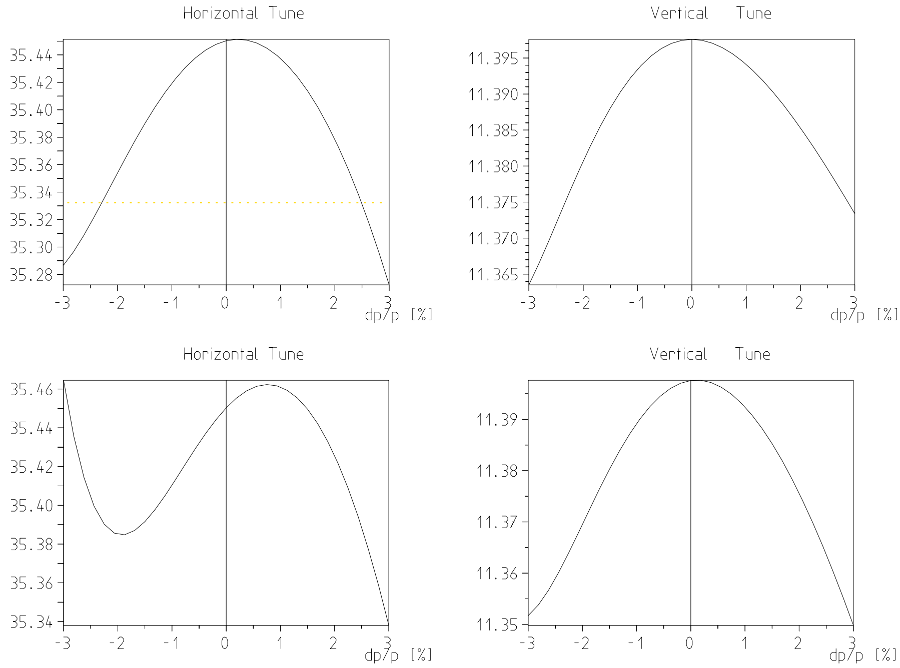


Figure 9: Top figure shows tunes vs. momentum after second order sextupole optimization. Second order chromaticities could be slightly reduced compared to fig. 7, but remain stubborn. Finally they are suppressed by additional octupoles in dispersive regions, as shown in the bottom figure: Small positive linear chromaticity was also used, to “fold” the horizontal tune vs. momentum curve in the most narrow intervall (note different scale).

applied, which attack in first order the second order sextupole terms. Such a scheme basically would set sextupoles based on eq. 8 for chromaticity correction and first order cancellation, and transfer the 13 (assuming symmetry, otherwise 21) second order terms as right hand side vectors to a $13 \times P$ linear system for P octupole families. Such a scheme might be required for future, more challenging light source storage rings. Already blindly playing with octupoles provides promising results as shown in figures 8 (right) and 9 (bottom).

6 Summary

Design of a modern source lattice must not proceed by first designing the linear lattice for optimum emittance, straight length etc., but has to take into accout the sextupole scheme right from the beginning in order to a) find suitable places (large dispersion, decoupled betas) for the chromaticity sextupoles abnd b) to exploit symmetry and periodicity for cancellation of at least some of the sextupolar resonance drive terms.

Restoring dynamic acceptances to values sufficient for injection and for providing reasonable beam life time requires installation of several families of “harmonic” sextupoles which mainly compensate adverse effects from the chromaticity sextupoles. Degeneration of the configuration has to be tested and avoided.

First and second order sextupole terms will be minimized by means of a suitable code through many trials with different weighting of the diverse quantities to be minimized. Solutions for the sextupole pattern obtained this way are only based on perturbation theory, thus validity for the required amplitudes has to tested by tracking dynamic apertures. This should also include misalignments, magnet errors, and other effects disturbing periodicity and symmetry in order to ensure robustness of the solution.

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