4. Multi-particle dynamics

- Liouville's theorem
  - Phase space conservation ▪ Deterministic Chaos
- Particle distributions
  - Moments ▪ Measurements ▪ Sigma-matrix
  - Emittance ▪ Courant-Snyder parameters
- Beam Transformation
  - Particle and beam transformation
  - Periodicity
  - Matching
  - The FODO cell
- Lattice imperfections
  - Resonances
  - Orbit distortion and correction
  - Chromaticity

Liouville's theorem

"The 6-d phase space occupied by a beam behaves like an incompressible liquid."

(i.e. can be deformed, twisted, stretched etc. but will never shrink or blow up)

valid for Hamiltonian systems only:

Joseph Liouville
1809 - 1882

Transformation by **symplectic map** $\mathcal{M}$

$$\vec{X}_1 = \mathcal{M}(\vec{X}_0) \qquad \vec{X} = (x, p_x, y, p_y, z, p_z)$$

Jacobian of map $\mathcal{M}$ = (local) **transfer matrix** $\boldsymbol{M} = \left\{ \frac{\partial \mathcal{M}}{\partial \vec{X}} \right\} \bigg|_0$

Symplecticity implies* $|M| = 1$

⇨ conservation of (local) phase space volume

*symplecticity is more than that: $n(n-1)/2$ constraints on matrix M (n = dimension)

Conservation of phase space volume

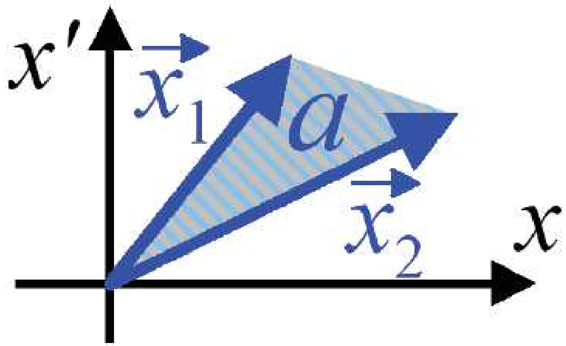
Linear beam transformation with transfermatrix: $\vec{x}_1 = M \cdot \vec{x}_o$

symplectic matrix, i.e $|M| = 1$

2-d phase space vectors $\vec{x}_1$, $\vec{x}_2$ enclose area $a = \frac{1}{2}|\vec{x}_1 \times \vec{x}_2|$

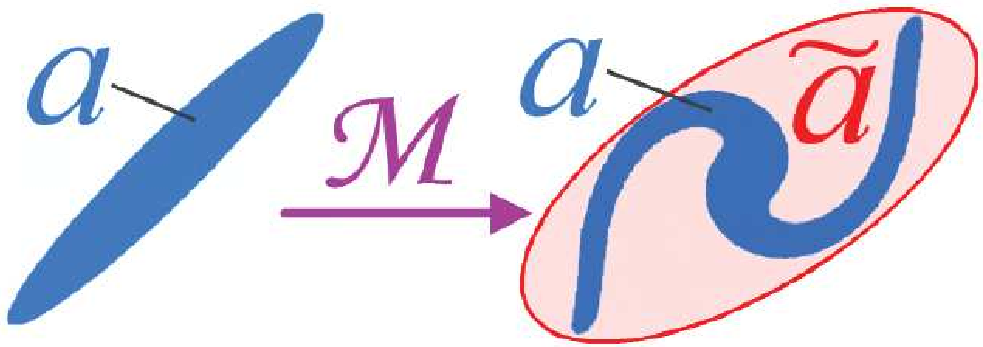
Linear transformation $\vec{x} = M\vec{x}_o \quad \longrightarrow$

Area $a = \frac{1}{2}|(M\vec{x}_{1o}) \times (M\vec{x}_{2o})| = \frac{1}{2}|M|\,|\vec{x}_{1o} \times \vec{x}_{2o}| = |M|a_o$



General transformation with non-linear *map*: $\vec{x}_1 = \mathcal{M}(\vec{x}_o)$

Symplectic map: local Jacobian $\left|\frac{d\mathcal{M}}{d\vec{x}}\right| = 1$



Filamentation

Area a is conserved, but area $\tilde{a}$ to accept the beam is increased.

$\rightarrow$ irreversibility!

Example: symplectic mapping

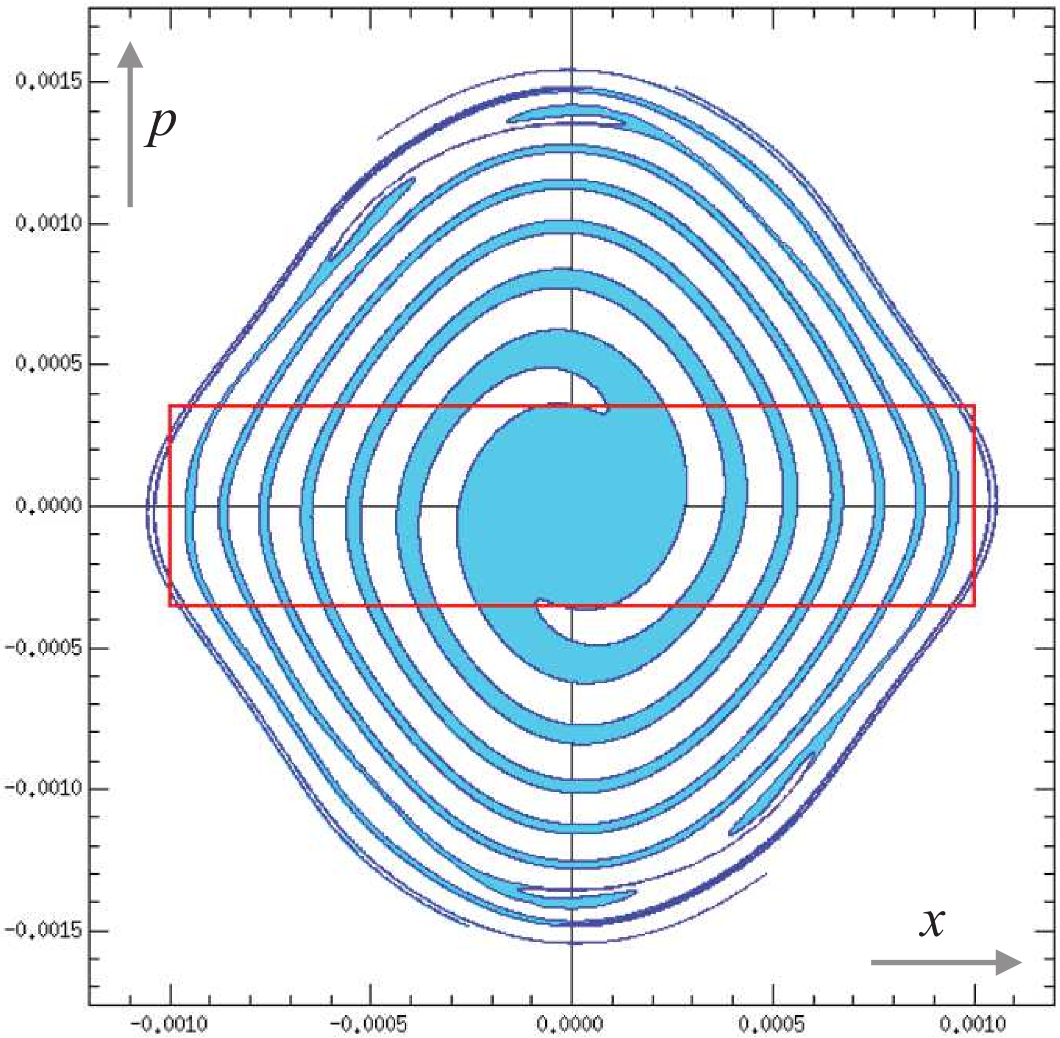
Nonlinear Hamiltonian:
$$H(x,p;s) = \frac{p^2}{2} - \frac{F(s)}{m}\cos(mx)$$

Equations of motion:
$$x' = \frac{\partial H}{\partial p} = p$$
$$p' = -\frac{\partial H}{\partial x} = -F(s)\sin(mx)$$

Assume: cell length d,
$F \neq 0$ in small region Δs at $s = d/2$

One-turn mapping:
$$x \leftarrow x + \tfrac{d}{2}p$$
$$p \leftarrow p - F\Delta s \sin(mx)$$
$$x \leftarrow x + \tfrac{d}{2}p$$

Symplectic: $\left|\frac{\partial\, \Delta x; \Delta p}{\partial\, x; p}\right| = 1 \;\rightarrow$ *prove!*



Test (10 recursions): initial / final area: Filamentation, but conservation of area.
(area measurements by IDL function `poly_area`: 1.41107e-06 1.41089e-06)

Chaos

Non-linear distortion of harmonic oscillation near $\psi = r/s$ ($r, s \in \mathrm{N}$, "small")

$\Rightarrow$ formation of $2s$ fix-points (**resonance**):

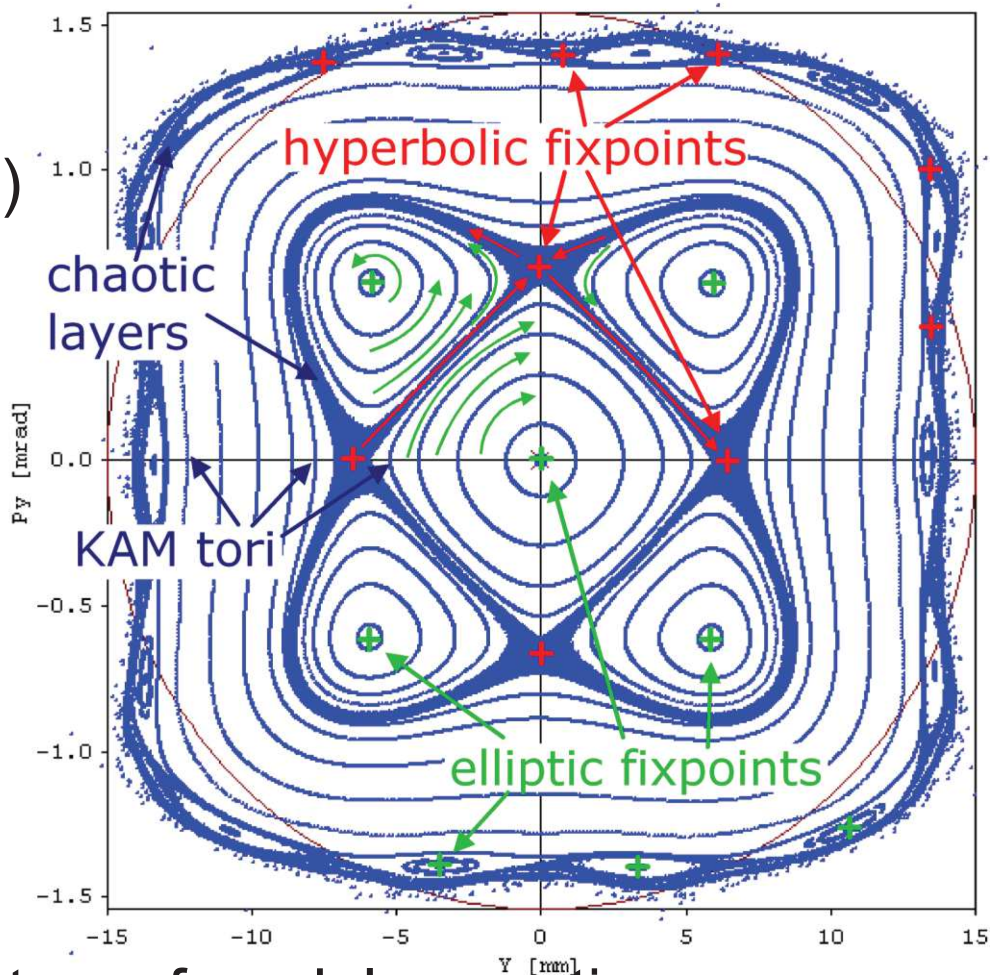
s **elliptic fix-points**: stable (μ complex)

$\Rightarrow$ s "islands"

s **hyperbolic fix-points**: unstable (μ real)

$\Rightarrow$ chaotic layers between islands.

$\Rightarrow$ Filamentation of phase space area due to area conservation and rapid increase of enclosing curve length.



Deterministic chaos

weak causality still valid:

same causes $\rightarrow$ same effects

strong causality invalid:

similar causes $\not\rightarrow$ similar effects

Self-similarity: islands become new centers of modulo-s-motion.

Storage ring design challenge: optimization of dynamic acceptance

(=central stable phase space area)

Particle distributions

- Particle beam (bunch) contains $\mathrm{N} = 10^6 ... 10^{12}$ particles.
  $\rightarrow$ continuous distribution
- phase space density = statistical distribution function $\rho(\vec{X})$
- Characterization by **moments**:
  - 0th $1 = \int \rho(\vec{X})\, d\vec{X}$ normalized to particle number N
  - 1st $\langle X_i \rangle = \int \rho(\vec{X})\, X_i\, d\vec{X}$ **beam orbit** $\bar{\vec{X}} = \{\langle X_i \rangle\}$
  - 2nd $\langle X_i X_j \rangle = \int \rho(\vec{X})\, X_i\, X_j\, d\vec{X}$ **Σ-matrix** $\Sigma = \{\langle X_i X_j \rangle\}$

⇨ 6 (r.m.s.) beam sizes $\sigma_i = \sqrt{\langle X_i^2 \rangle}$

⇨ 15 correlations $\langle X_i X_j \rangle,\ i \neq j$

= 21 independent elements

centered Σ-matrix: $\Sigma = \{\langle (X_i - \bar{X}_i) \cdot (X_j - \bar{X}_j) \rangle\}$

Measurements

0th moment

current transformer

beam current $I(t)$
beam charge $Q = qN = \int I \, \mathrm{d}t$

1st moment

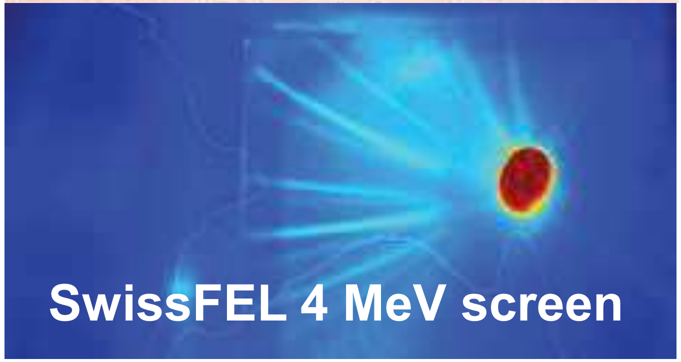
beam position monitor

transverse beam positions $\bar{x}, \bar{y}$

2nd moment

fluorescent screen + camera

beam sizes σ_x, σ_y
and tilt angle $\tan 2\vartheta = \dfrac{\langle xy \rangle}{\sigma_x^2 - \sigma_y^2}$



The Σ-matrix

Linear beam transformation by matrix M

- 0^{th} moment: invariant (not assuming beam losses)
- 1^{st} moment: orbit $\vec{\bar{X}}_1 = M\,\vec{\bar{X}}_0$
- 2^{nd} moment: Σ-matrix $\boxed{\Sigma_1 = M\Sigma_0 M^T}$ *(prove!)*

Symplecticity: $|M| = 1 \quad \Rightarrow \quad |\Sigma_1| = |\Sigma_0| =$ constant.

|Σ| is invariant (in *linear* dynamics)

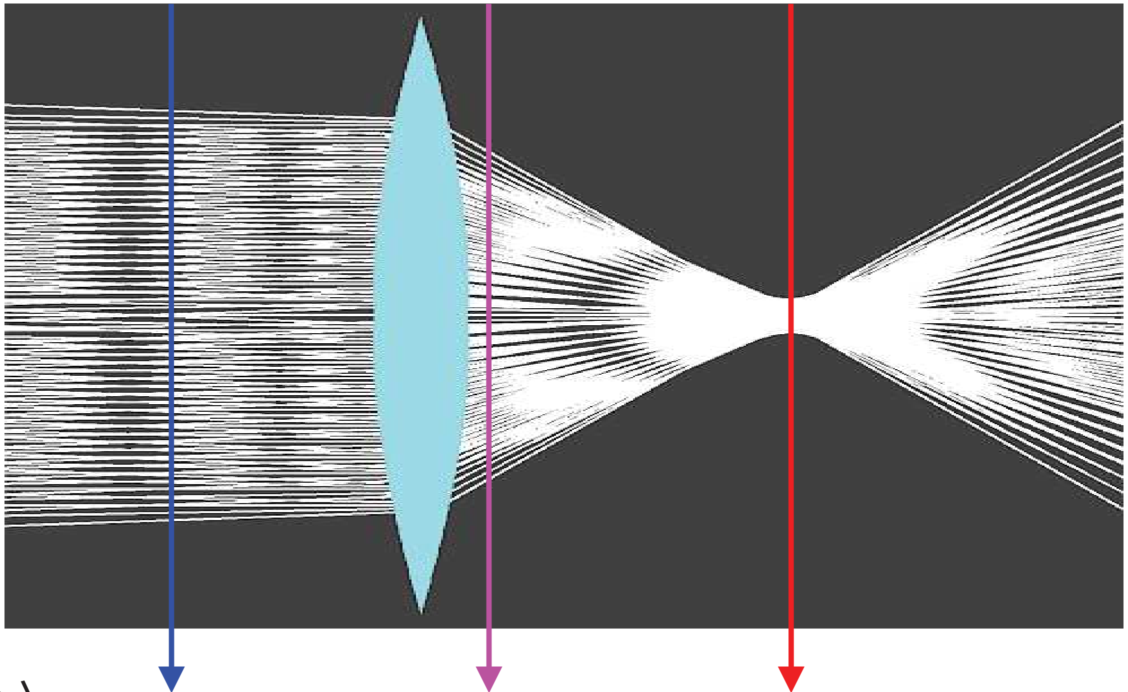
Example
$(n = 2)$

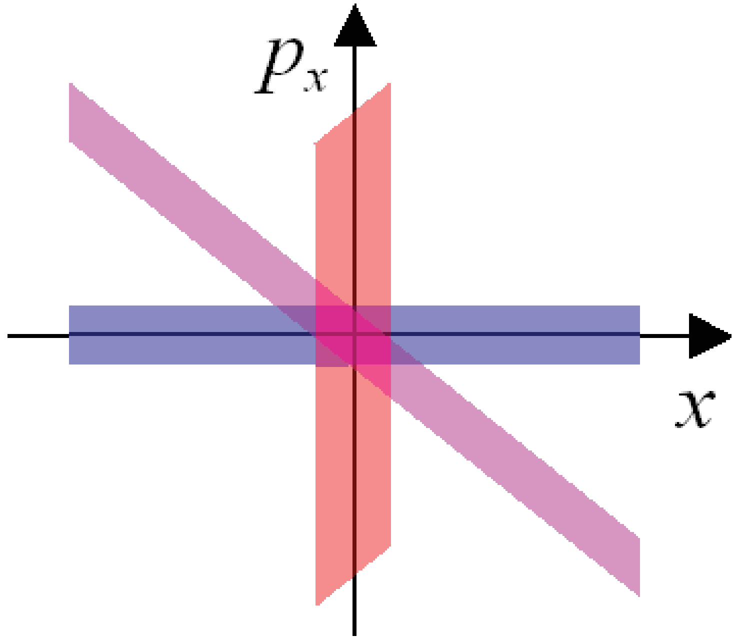
$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix}$$

$$|\Sigma| = \langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 = \text{constant} := \varepsilon^2$$

ε = **emittance ,** invariant of motion (in 2-d)

Example: focusing a beam





$\langle x^2 \rangle$	large	large	small
$\langle p_x^2 \rangle$	small	large	large
$\langle x p_x \rangle$	$\mathbf{0}$	large	$\approx \mathbf{0}$
ε	- - - - - - - constant - - - - - - - -		

Phase space view

Emittance is the invariant phase space area (2-d).

General linear dynamics:

$\sqrt{|\Sigma|}$ is the invariant 6-d phase space volume.

Transverse emittance definitions

- normalized (true) emittance ε_{xn} in $\{x, p_x\}$ plane, resp. ε_{yn}
- geometric emittance ε_x in $\{x, x'\}$ plane

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

  - often used, in analogy to ray optics: $x' = p_x/p << 1$
  - but x' is not the canonical conjugate to x!
  - $\varepsilon_{xn} \approx < \beta\gamma > m_o c\ \varepsilon_x$ if $x' << 1$ paraxial beam
    and $|p - p_0| << p_0$ monochromatic beam

- Emittance units
  - $[\varepsilon] = \mathrm{m}[\cdot\mathrm{rad}]$ or $\mathrm{mm}\cdot\mathrm{mrad}$ → storage ring community: $\mathrm{nm}[\cdot\mathrm{rad}]$
  - $[\varepsilon_n] = m_o c\ \mathrm{mm}\cdot\mathrm{mrad}$ → linac community: $\pi\ \mathrm{mm}\cdot\mathrm{mrad}$

Longitudinal emittance definitions

- relative to reference particle at $(s(t), p_0)$ in $\{\Delta s, \Delta p\}$ plane

$$\varepsilon_s = \sqrt{\langle \Delta s^2 \rangle \langle \Delta p^2 \rangle - \langle \Delta s \Delta p \rangle^2}$$

- alternative coordinates
  - normalized momentum $\Delta p \rightarrow \delta = \Delta p / p_0$
  - energy $\Delta p \rightarrow \Delta E = \Delta p \ \beta_0 c$
  - time $\Delta s \rightarrow \Delta t = \Delta s / \beta_0 c$
  - phase $\Delta s \rightarrow \Delta \phi = 2\pi \ \Delta s / \lambda$ with λ RF wavelength.
- "longitudinal emittance" uncommon term in synchrotrons
  - little correlation $<\Delta s \Delta p> \approx 0$
  - use bunch length σ_s and relative momentum spread σ_δ

Courant-Snyder parameters

transverse 2-d phase space using $\{x, x'\}$ $\varepsilon = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$

- Normalized Σ-matrix $\boldsymbol{\sigma}$

$$\sigma = \frac{\Sigma}{\sqrt[2]{|\Sigma|}} = \frac{\Sigma}{\varepsilon} = \begin{pmatrix} \frac{\langle x^2\rangle}{\varepsilon} & \frac{\langle xx'\rangle}{\varepsilon} \\ \frac{\langle xx'\rangle}{\varepsilon} & \frac{\langle x'^2\rangle}{\varepsilon} \end{pmatrix} := \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

E. D. Courant & H. S. Snyder, *Theory of the Alternating Gradient Synchrotron*, Annals of physics 3, 1958

don't mix with the relativistic parameters β, γ!

- $\boldsymbol{\alpha, \beta, \gamma}$ Courant-Snyder parameters $|\sigma| = \beta\gamma - \alpha^2 = 1$
- Beta function $\beta\ [\mathrm{m}] = \sigma_x^2/\varepsilon \quad [\ \alpha = -\beta'/2\ ,\ \ \gamma = (1+\alpha^2)/\beta\ ]$
- $\boldsymbol{\Sigma = \varepsilon\sigma}$ characterizes the physical beam: *"the body"*
  - ε defines "mass" of beam (invariant area in phase space): *"the flesh"*
  - σ defines "shape" of beam (variable orientation in phase space): *"the bones"*

Beam transformation

- Transfer matrix
  - piecewise constant elements (drift, quadrupole etc.)
  - element matrix = part of an harmonic oscillation $x'' + kx = 0$
  - lattice matrix = product of the element matrices $M = \prod_{i=1}^{N} M_{N-i+1}$
- Single particle transformation $\vec{x}_1 = M\,\vec{x}_0$
- Beam description by Σ-matrix

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix} = \sigma\varepsilon, \quad \sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \quad \varepsilon = \sqrt{|\Sigma|}, \quad |\sigma| = 1$$

- Beam transformation

$$\sigma_1 = M\,\sigma_0\,M^T$$

or explicitly:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1 = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \cdot \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

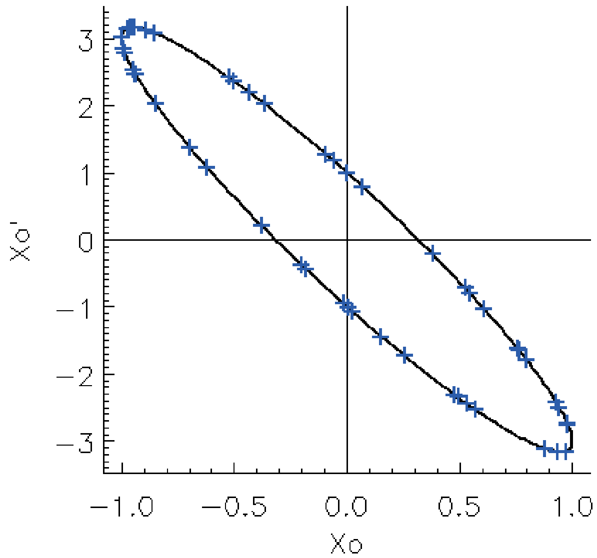
Example: drift space

Transfer matrix

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Initial beam ellipse

$\beta_o = 1$ [m], $\alpha_o = 3$
50 particles $(x, x')_{io}$ at amplitude $2J = 1$ [m·rad]

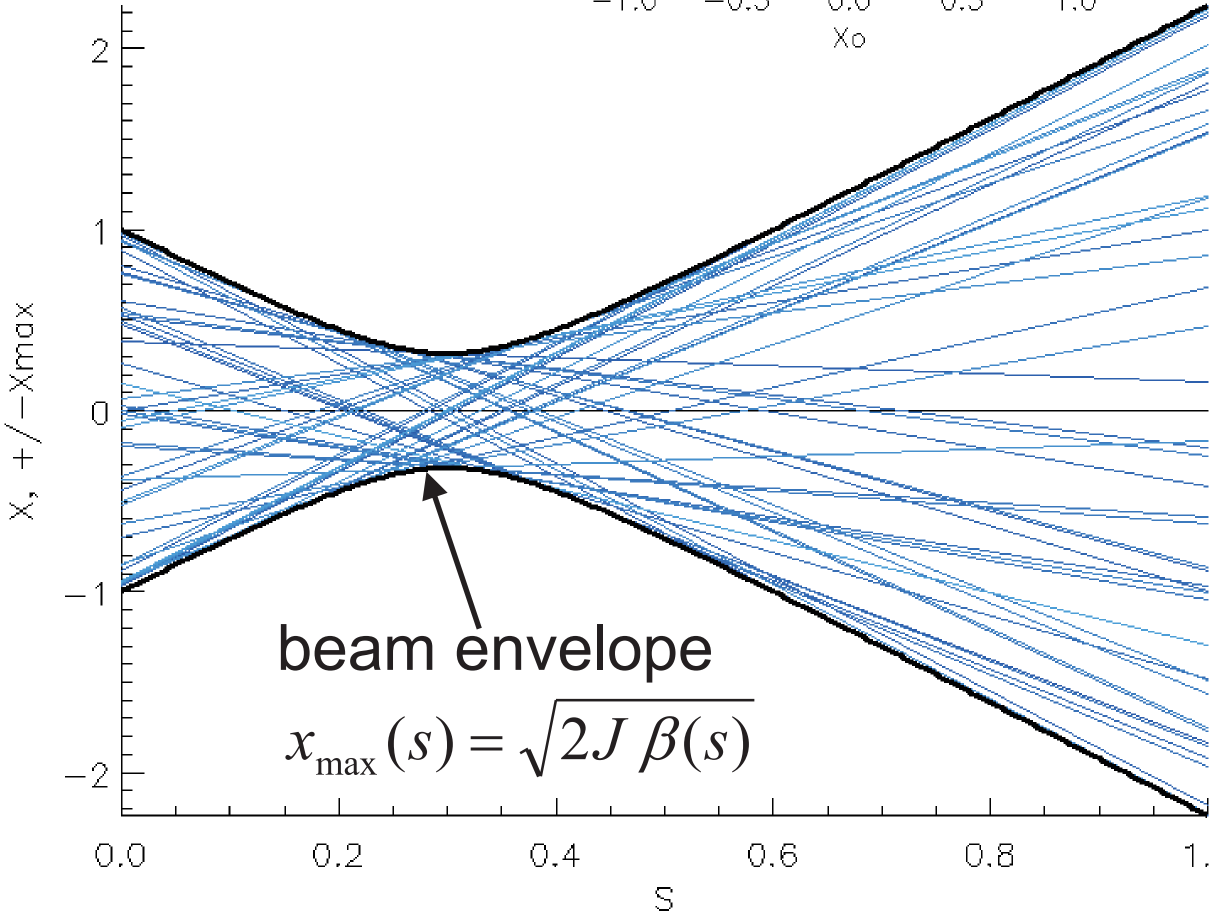


Single particle transformation:

$$x_i(s) = x_{i0} + s\, x'_{i0}$$

CS parameter (σ–matrix) transformation:

$$\beta(s) = \beta_0 - 2s\alpha_0 + s^2\gamma_0$$
$$\alpha(s) = \alpha_0 - s\gamma_0$$
$$\gamma(s) = \gamma_0$$



Periodicity

- the sigma matrix $\sigma(s)$ (resp. $\beta(s)$) tells how a beam propagates from initial conditions σ_0
- periodic structure may have periodic solution $\sigma_{\text{per}} = M\sigma_{\text{per}}M^T$
- the periodic solution σ_{per} is a pure *lattice* property, independent of the initial beam conditions σ_0.
- Calculation:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_1 = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \cdot \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0 \quad \Rightarrow \quad \beta = \frac{m_{12}}{\sqrt{1-[\frac{1}{2}(m_{11}+m_{22})]^2}} \qquad \alpha = \frac{m_{11}-m_{22}}{2m_{12}}\beta \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

- periodicity condition: $|\tfrac{1}{2}\mathrm{Tr}(M)| = \tfrac{1}{2}|m_{11} + m_{22}| < 1$
otherwise no periodic solution exists and β, α, γ are undefined.
- Betatron tune Q (from slide 86):
$|\cos(2\pi Q)| < 1$ with $\cos\mu = \tfrac{1}{2}\mathrm{Tr}(M)$

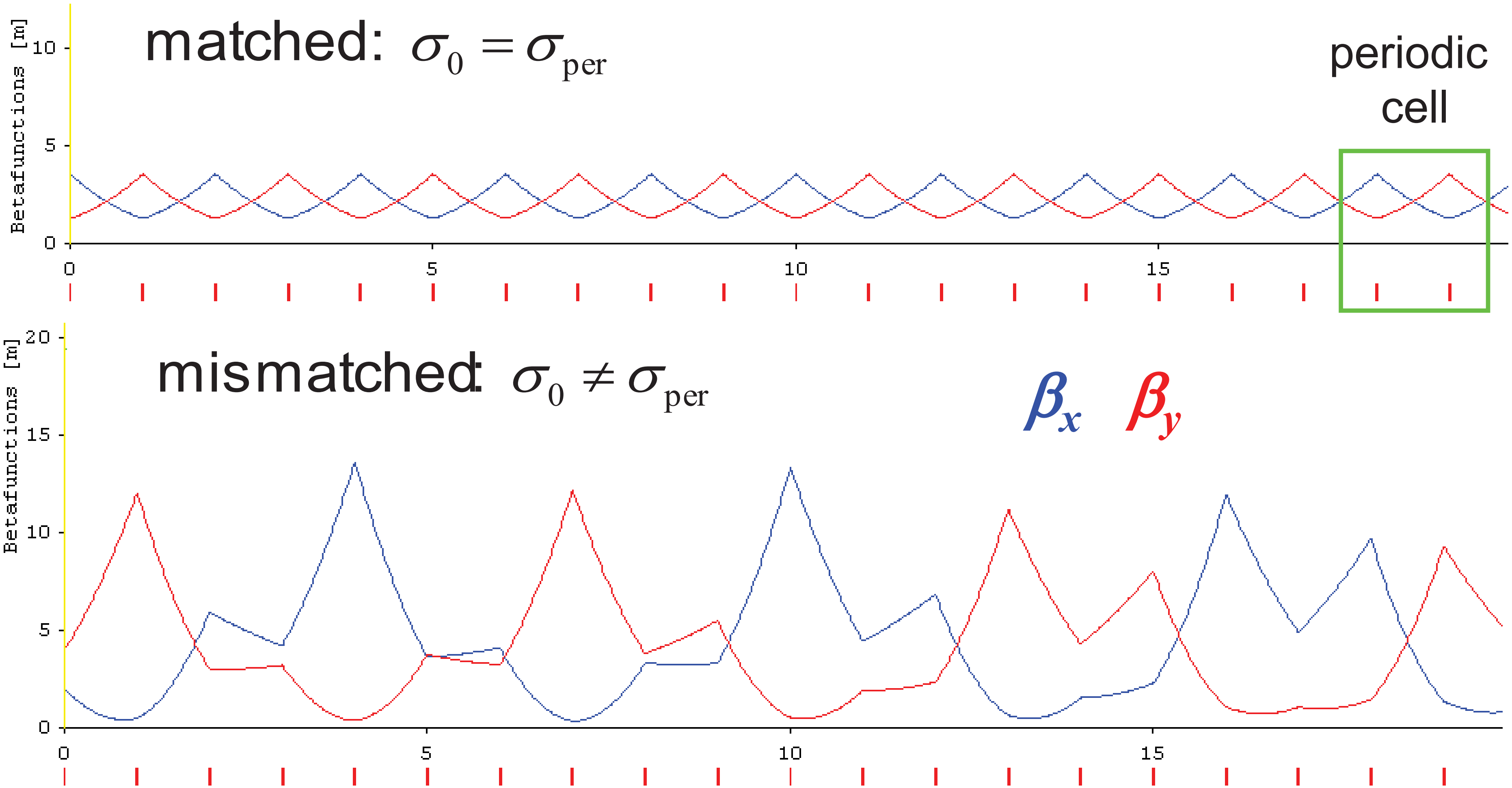
$$\Rightarrow \quad \beta = \frac{m_{12}}{\sin 2\pi Q} \qquad \alpha = \frac{m_{11}-m_{22}}{2\sin 2\pi Q} \qquad \gamma = -\frac{m_{21}}{\sin 2\pi Q}$$

- Matching: adjust $\sigma_0 = \sigma_{\text{per}}$

Matching

Example: FODO line

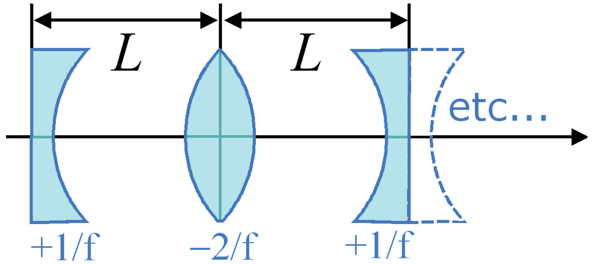
alternating horizontal and vertically focusing quadrupoles



Note: in electron/positron storage rings, the beam matches itself within milli-seconds through synchrotron radiation

Application: the FODO cell

FODO = Focusing - space - Defocusing - space
= repetition of the quadrupole doublet



choose symmetry point ($\alpha = 0$)and define $r := \frac{L}{f}$

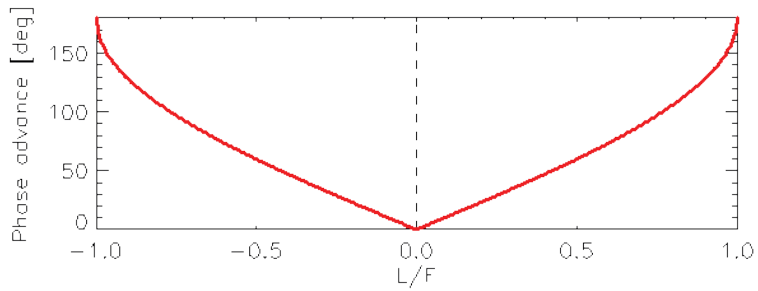
$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2/f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} =$$

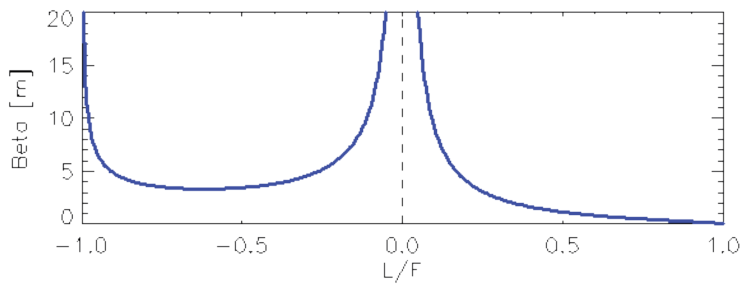
$$\begin{pmatrix} 1-2r^2 & 2L(1-r) \\ -2r(1+r)/f & 1-2r^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$\cos\mu = \frac{m_{11}+m_{22}}{2} = 1 - 2r^2$
$\longrightarrow$ stability for $0 < |r| < 1$

$\sin\mu = 2r\sqrt{1-r^2}$
$\longrightarrow \quad \beta = \frac{m_{12}}{\sin\mu} = |f|\sqrt{\frac{1-r}{1+r}}$

$\alpha = \frac{m_{11}-m_{22}}{2\sin\mu} = 0 \rightarrow \gamma = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta}$

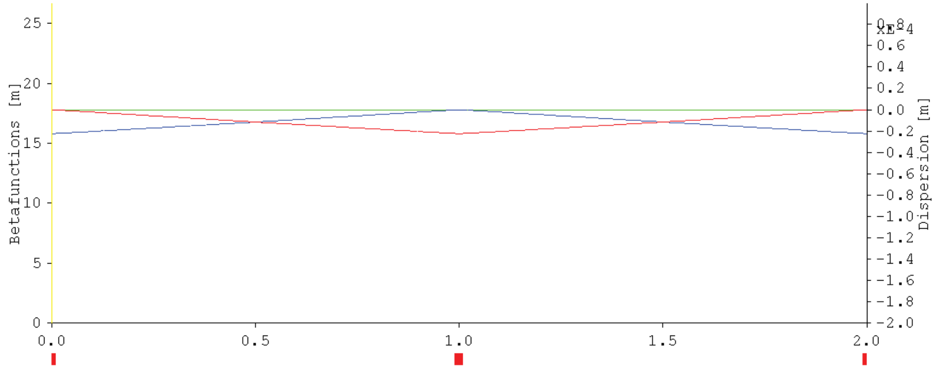




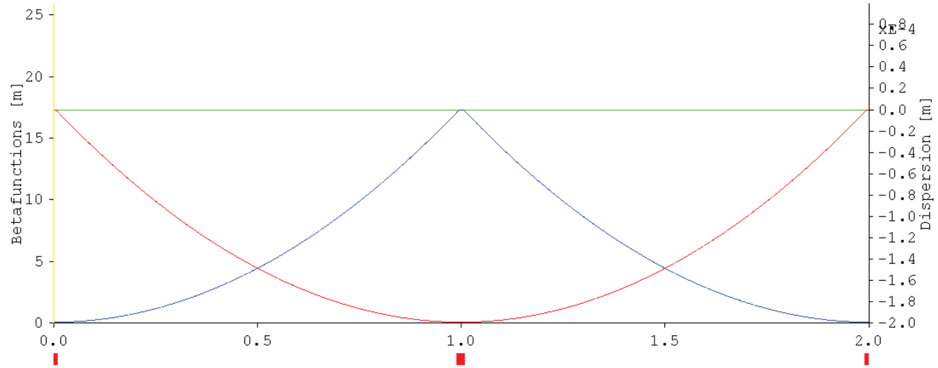
FODO cell beta functions

horizontal and vertical beta functions in a FODO-cell.

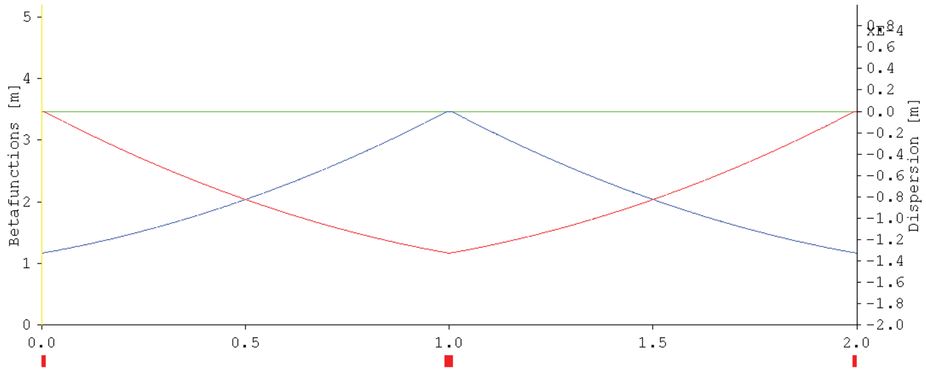
$L/F = 0.06 \;\rightarrow\; Q_x = Q_y = 0.02$



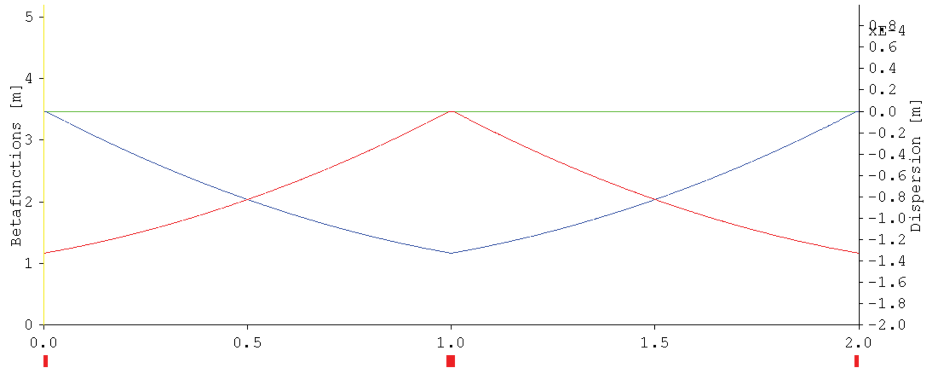
$L/F = 0.99 \;\rightarrow\; Q_x = Q_y = 0.46$



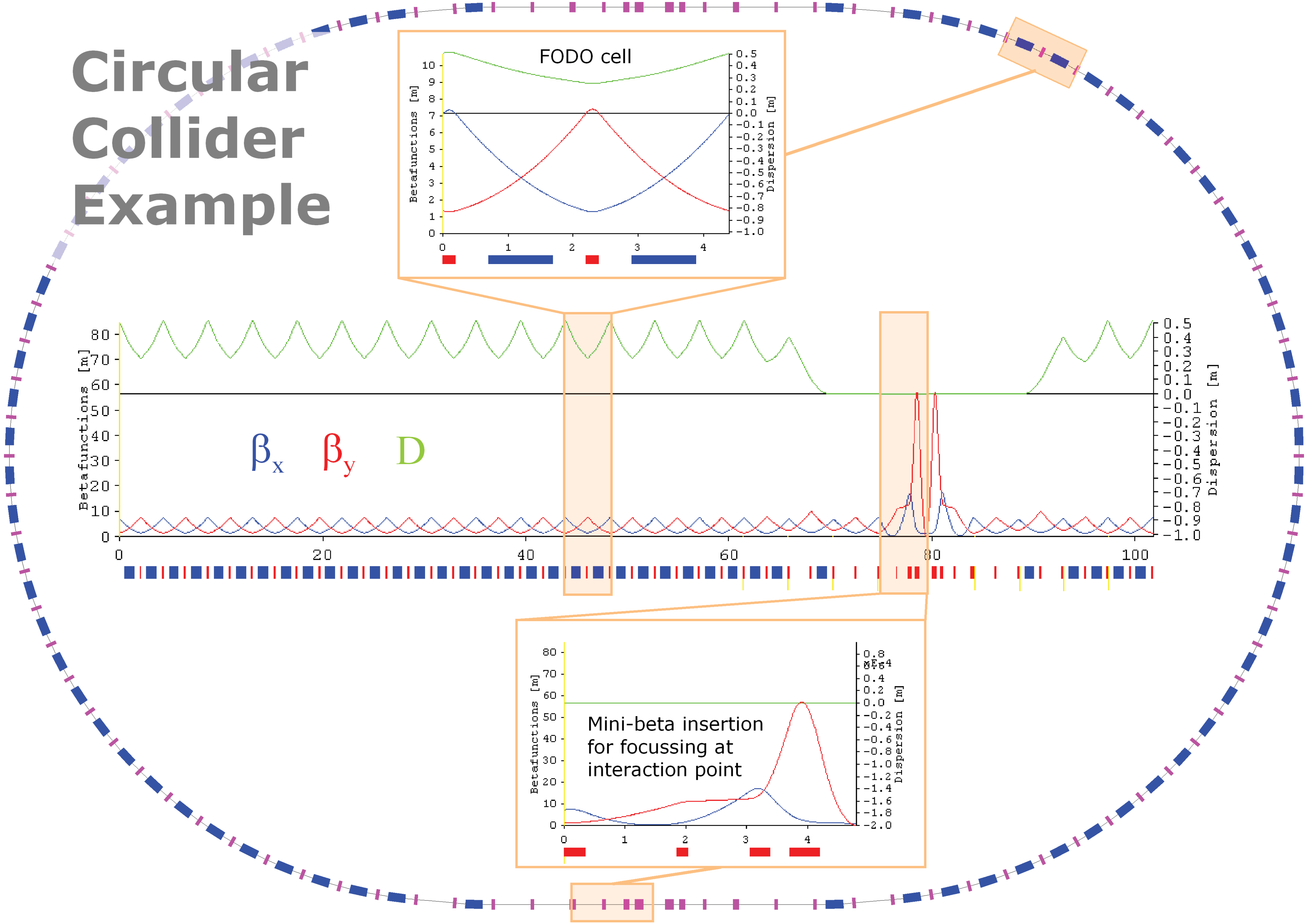
$L/F = 0.5 \;\rightarrow\; Q_x = Q_y = 0.17$



$L/F = -0.5 \;\rightarrow\; Q_x = Q_y = 0.17$



Circular Collider Example



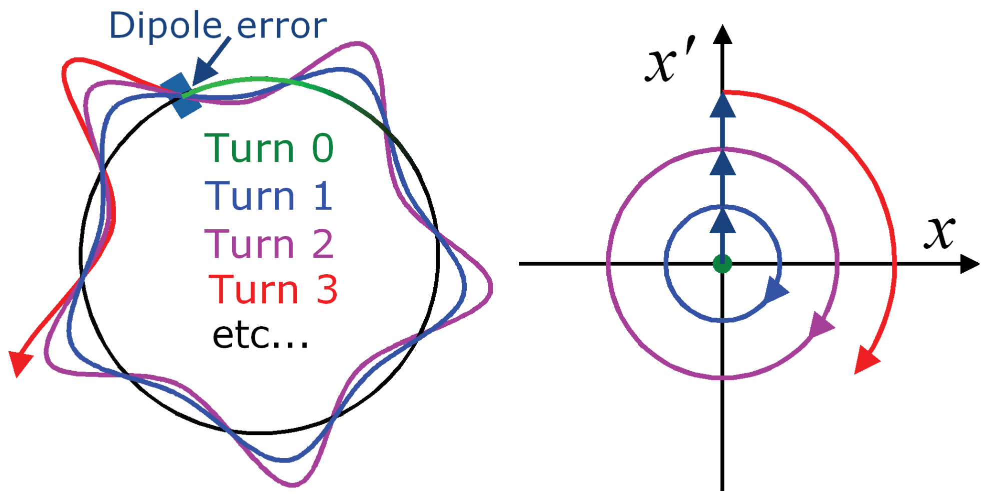
Resonances

Dipole error $\rightarrow$ kick on beam:

$\Delta x' = \int b_1 ds = \frac{\int B_y\, ds}{(B\rho)}$

$\rightarrow$ Increase of betatron amplitude if tune Q_x near integer number $\rightarrow$ beam loss!

(stability near half integer, alternating kicks $+ - + -$ ...)



Quadrupole error: $\Delta x' = x \cdot \int b_2 ds$
$\longrightarrow$ unstable if tune Q near half integer!

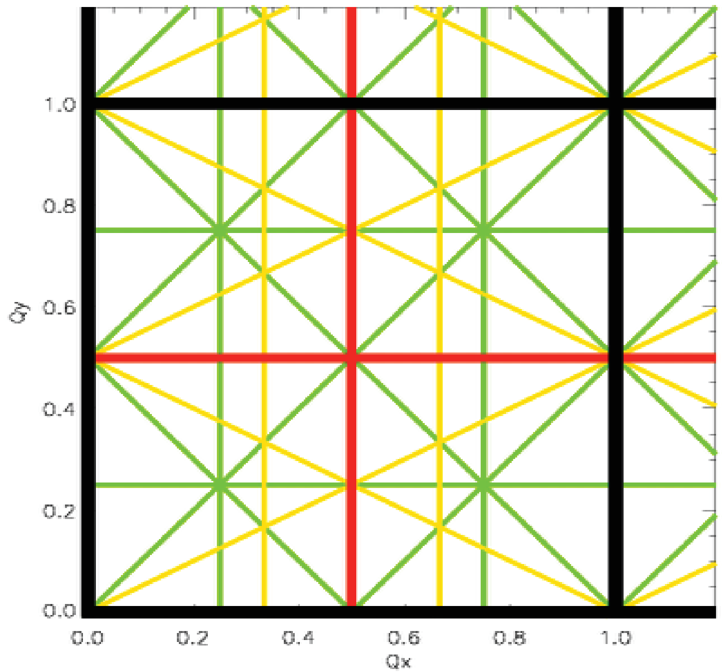
Multipoles drive any resonance:
$AQ_x + BQ_y = C \quad (A, B, C$ integers)

Resonance order $n = |A| + |B|$

B even / odd $\longleftrightarrow$ regular (b_n) / skew (a_n) multipoles

Tune diagram:

1 Dipole
2 Quadrupole
3 Sextupole
4 Octupole

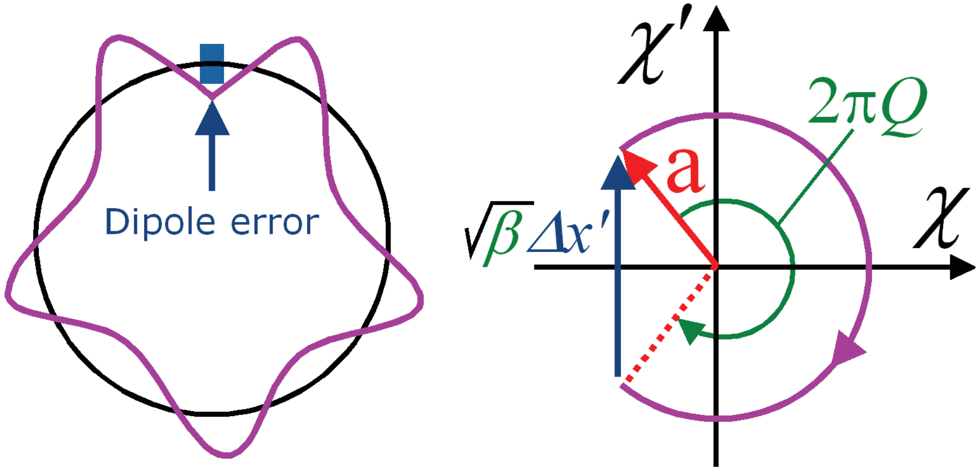


Orbit distortion

Dipole kick $\Delta x'$ and *non*-integer tune
$\longrightarrow$ perturbation of periodic orbit

use normalized coordinates:
$\Delta\chi' = \sqrt{\beta}\Delta x' \quad \longrightarrow \quad \sin\left(\pi - \frac{1}{2}2\pi Q\right) = \frac{\Delta\chi'/2}{a}$:

Orbit amplitude $\quad a = \dfrac{\sqrt{\beta}\,\Delta x'}{2\sin\pi Q}$



Kick at location k, observation at location i $\quad\longrightarrow\quad$ *closed orbit* equation:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_i = M_{k\to i}\left[\begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_k + M_{i\to k}\begin{pmatrix} x \\ x' \end{pmatrix}_i\right]$$

$\longrightarrow$ use normalized coordinates (transformations are just rotations)
$\longrightarrow$ generalize to m kicks (superposition of linear solutions)

$$x_i = \sum_{k=1}^{m} \frac{\sqrt{\beta_i\beta_k}}{2\sin\pi Q}\cos(\phi_{ki})\,\Delta x'_k \qquad \phi_{ki} = \begin{cases} \phi_i - \phi_k + \pi Q & (i < k) \\ \phi_i - \phi_k - \pi Q & (i > k) \end{cases}$$

Orbit correction

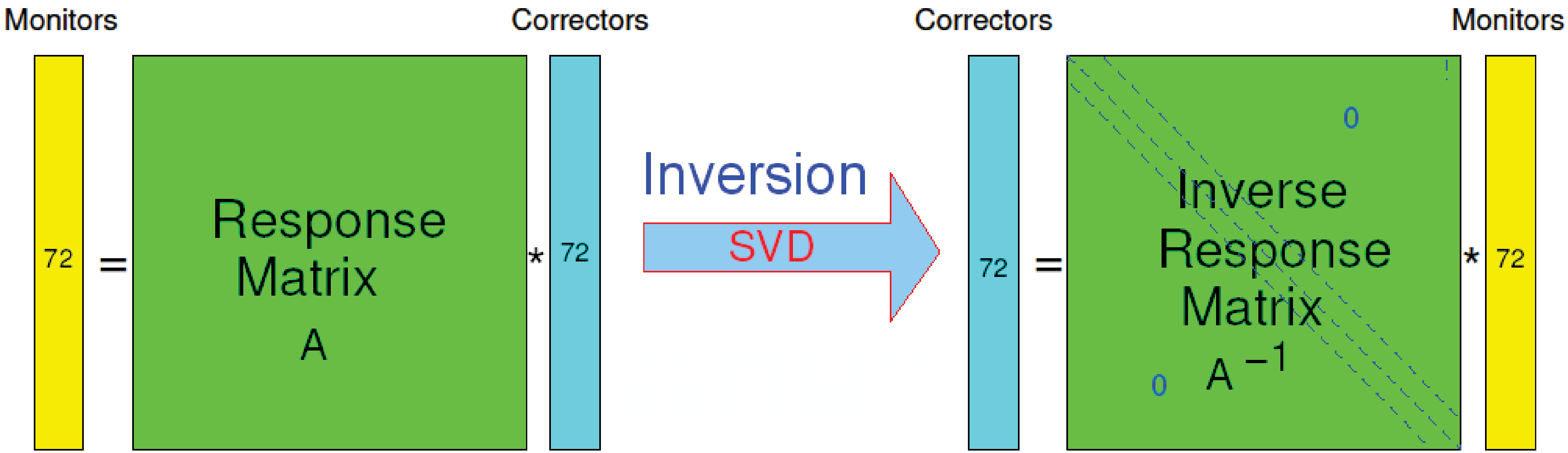
Orbit correction system:

m corrector magnets (small dipoles) and n beam position monitors (BPM).

Calculate (or measure) the $(m \times n)$ *Response Matrix* A:

element A_{ki} contains orbit at BPM i for single kick from corrector k:

(Example for $n = m = 72$, Ref.: M.Böge, *Orbit Feedback at SLS*, Cern Accelerator School Brunnen 2003)



Corrector settings for zero orbit obtained by $\{\Delta x'_k\} = -R^{-1} \cdot \{x_i\}$,

with vectors $\{x_i\}$ of n BPM orbit measurements, $\{\Delta x'_k\}$ of m corrector kicks.

Chromaticity

Quadrupole:

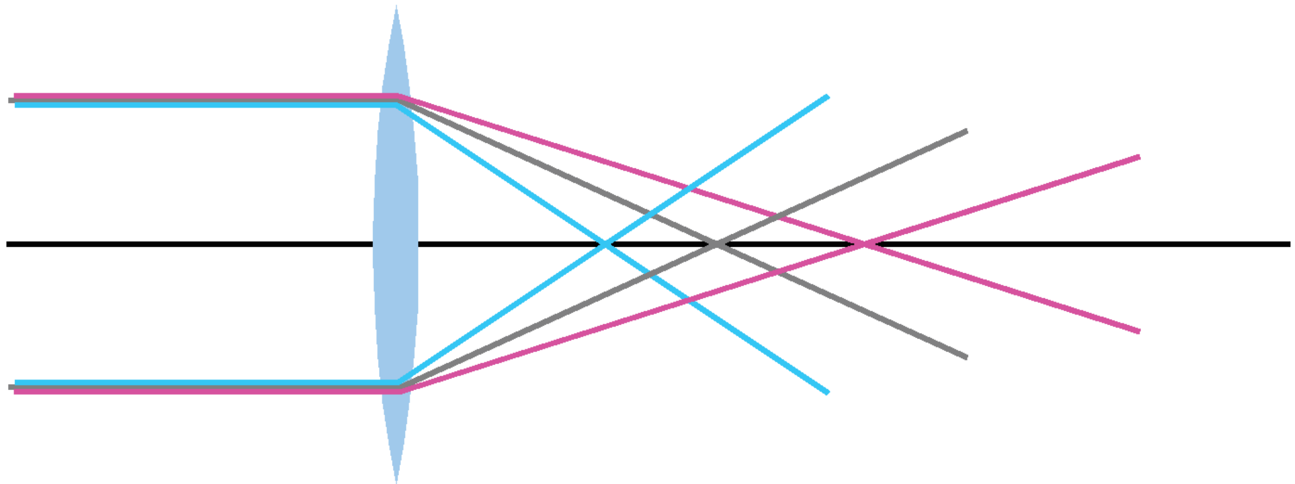
Length L

Strength $b_2 = \frac{1}{(B\,\rho)}\frac{d\,B_y}{d\,x}$

$(B\,\rho) := \frac{p}{e} = 3.3356\,\text{Tm} \cdot E[\text{GeV}]$

Kicks on particle: $\Delta x' = -b_2 L x \quad \Delta y' = b_2 L y \qquad (b_2 > 0 \rightarrow \text{horiz.foc.})$

Chromatic aberration: $b_2(\delta) = \frac{b_2}{(1+\delta)} \approx b_2\,(1-\delta) \qquad \delta := \frac{\Delta p}{p}$



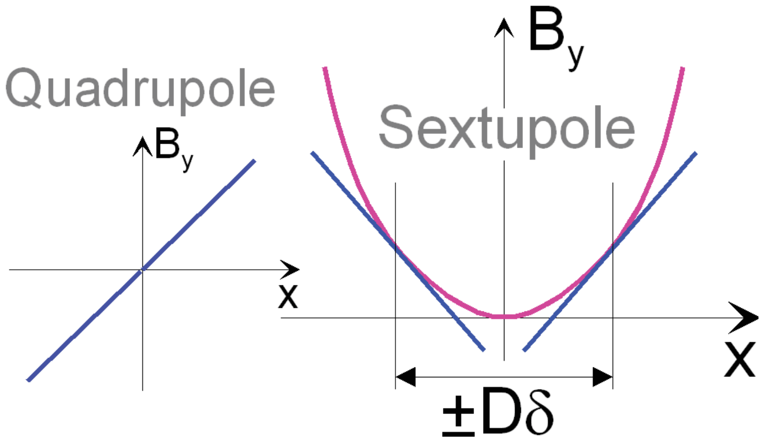
$\delta > 0$
$\delta = 0$
$\delta < 0$

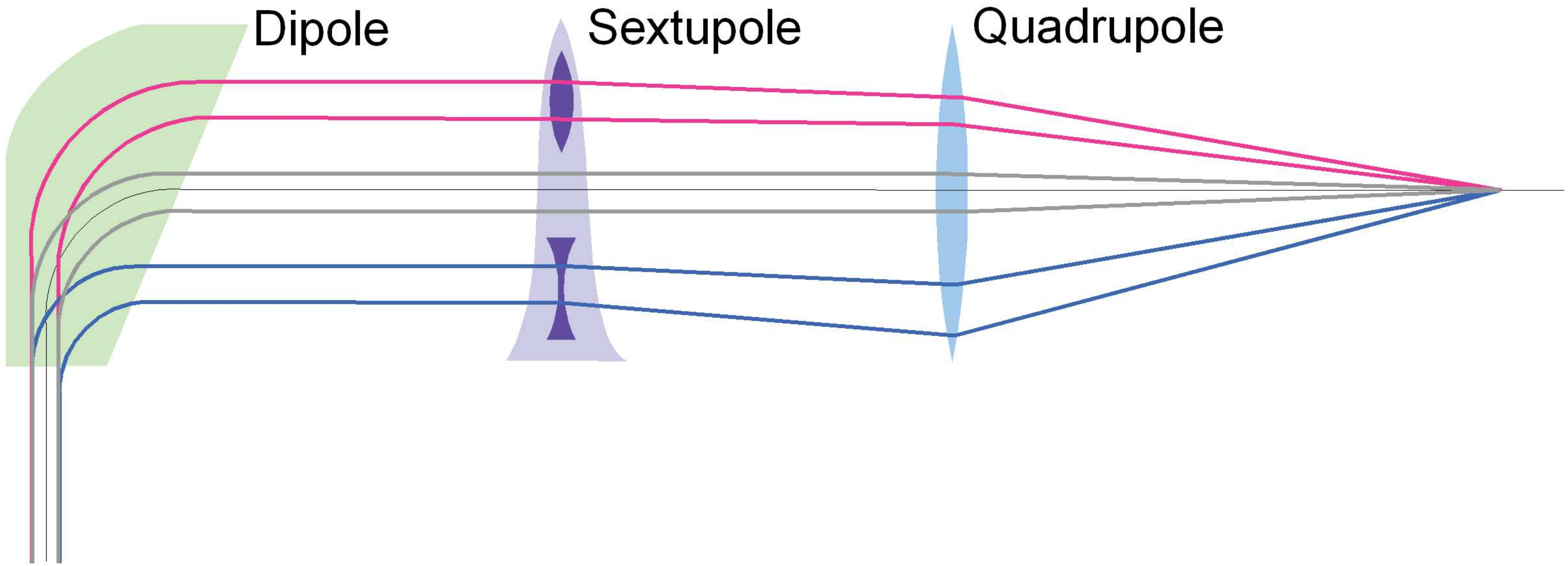
Chromaticity correction

Sextupole: $B_y(x) = \frac{1}{2}B''\,x^2$

local gradient: $B'_y(x) = B''\,x$

"Order" by momentum: $x(\delta) = D\delta$





Quadrupole: $b_2 = \frac{1}{(B\,\rho)}\frac{d\,B_y}{d\,x}$ **Sextupole:** $b_3 = \frac{1}{2}\frac{1}{(B\,\rho)}\frac{d^2\,B_y}{d\,x^2}$

$$\begin{aligned}\Delta x' &= -b_2 L x \\ \Delta y' &= b_2 L y\end{aligned} \qquad \begin{aligned}\Delta x' &= -b_3 L (x^2 - y^2) \\ \Delta y' &= 2 b_3 L x y\end{aligned}$$

Chromatic aberrations: $b_n(\delta) = b_n/(1+\delta) \approx b_n(1-\delta)$

Sextupoles in dispersive regions: $x \to D\delta + x \quad y \to y$

Kicks on a particle (keep up to second order in products of x, y, δ) :

Quadrupole: $\Delta x' = -b_2 L x + [b_2 L]\,\delta\,x \qquad \Delta y' = +b_2 L y - [b_2 L]\,\delta\,y$

Sextupole:
$$\begin{aligned}\Delta x' &= -[2 b_3 L D]\,\delta\,x - b_3 L (x^2 - y^2) - b_3 L D^2 \delta^2 \\ \Delta y' &= +[2 b_3 L D]\,\delta\,y + 2 b_3 L\,x y\end{aligned}$$

$\Longrightarrow$ Chromaticity correction for $(2 b_3 L D \stackrel{!}{=} b_2 L)$:

$$\xi_{x/y} = \pm\frac{1}{4\pi}\oint_C \left[2 b_3(s) D(s) - b_2(s)\right] \beta_{x/y}(s)\,ds \stackrel{!}{=} 0$$

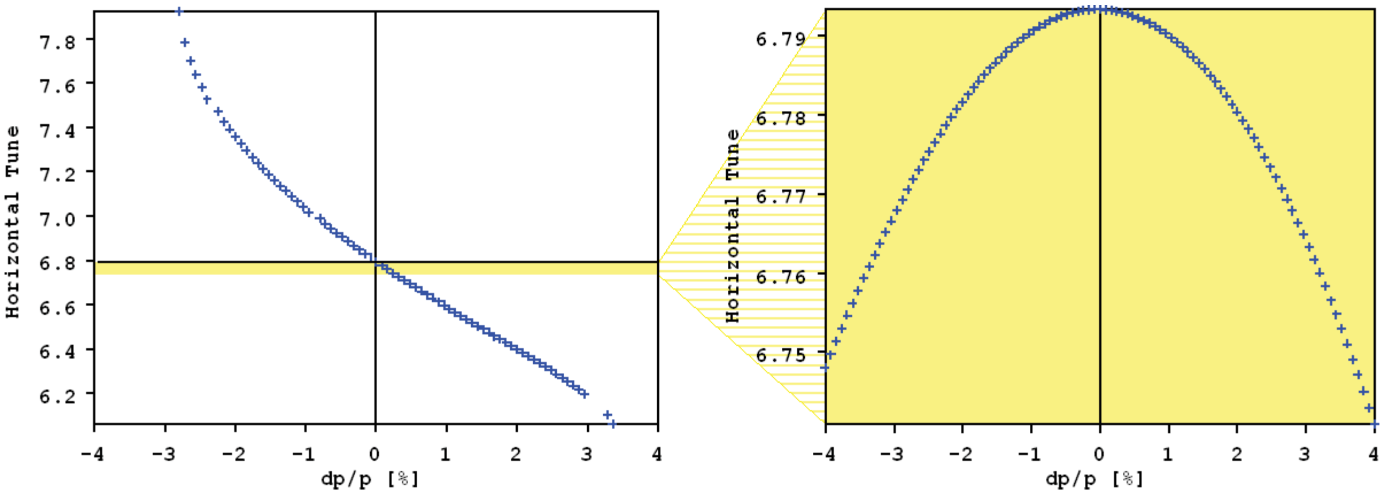
$\Longrightarrow$ nonlinear kicks... $\to$ Chaos, restriction of dynamic acceptance

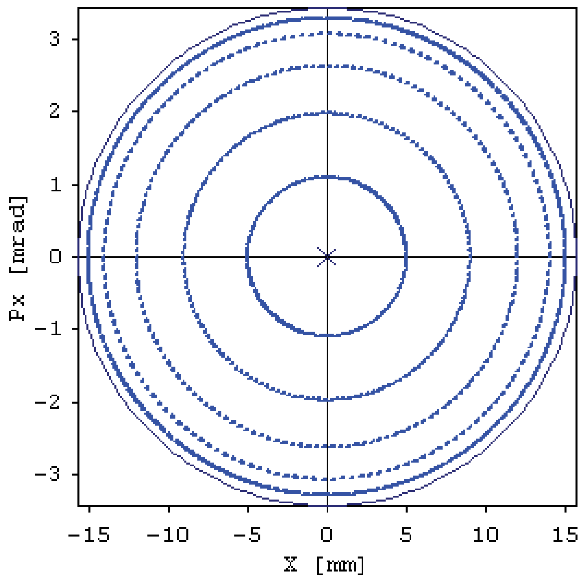
Chromaticity correction in the SLS

Horizontal betatron tune Q_x vs. $\Delta p/p$ for one period (=1/3 of the SLS lattice)

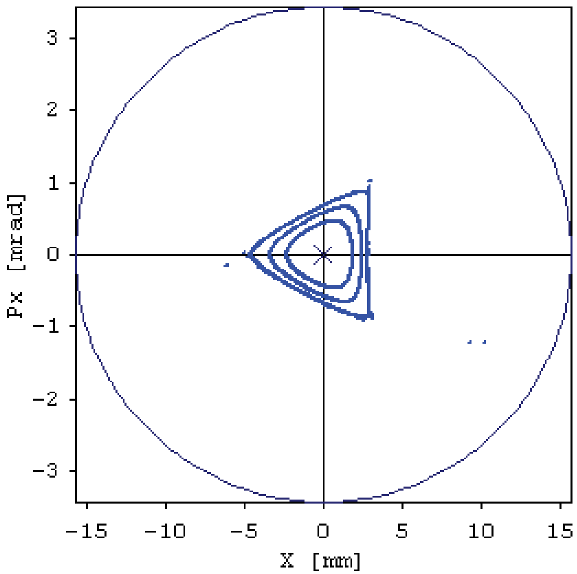
before →, and after →→ chromaticity correction.

Motion in horizontal phase space:

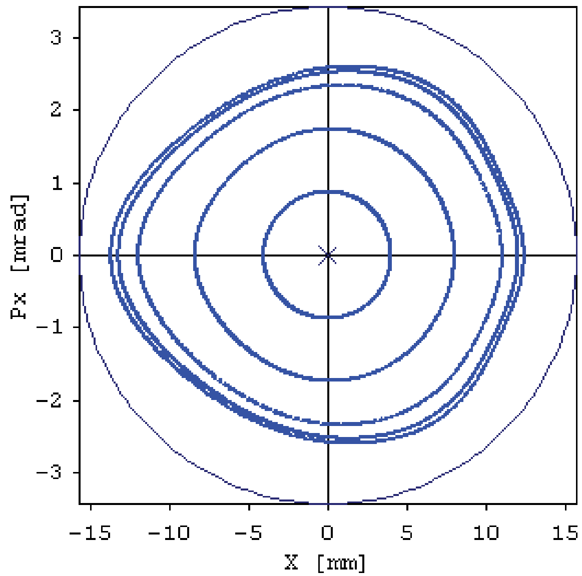




Linear oscillation before correction (no sextupoles).



Dynamic aperture breakdown due to sextupole non-linearity after straight-forward correction.



Partial restoration of dynamic aperture after careful distribution of sextupoles.