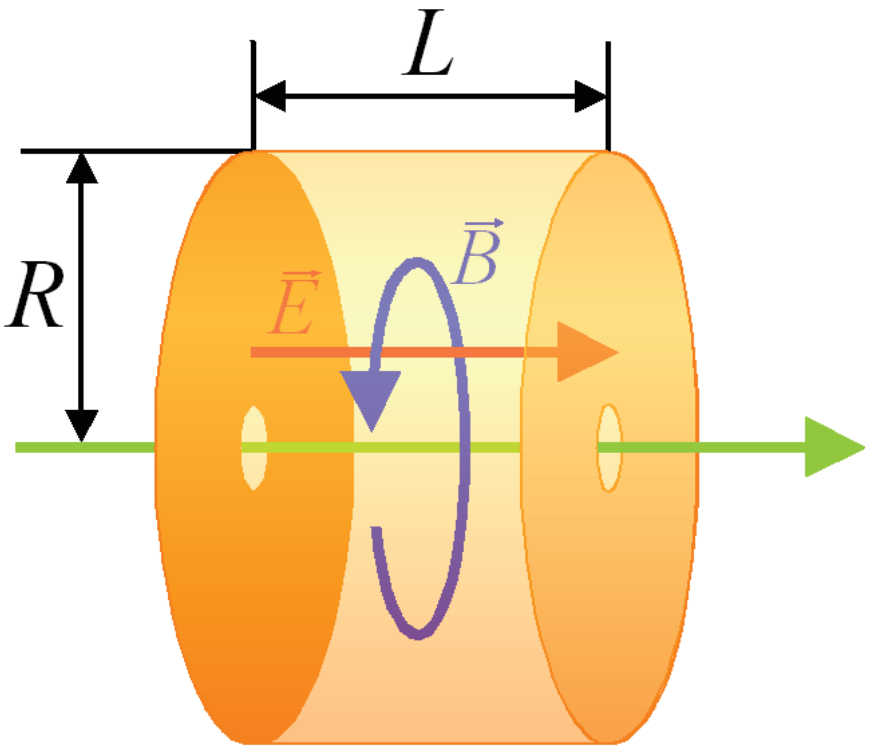
5. Longitudinal dynamics

for synchrotrons

- Acceleration
  - The pillbox cavity
- Synchrotron oscillations
  - Momentum compaction
  - Phase stability
  - Energy spread and bunch length
- Longitudinal acceptance
  - The bucket
  - Phase acceptance
  - Momentum acceptance

Acceleration



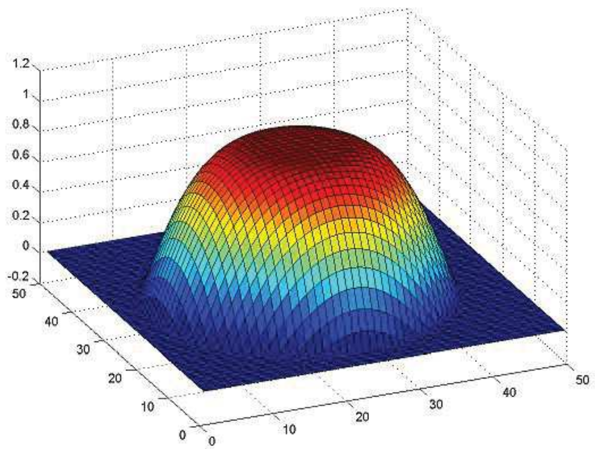
- Radio frequency (RF) cavity
  - metallic volume, vacuum
  - standing electromagnetic waves
  - spatial pattern (mode) determined by cavity geometry
- Pillbox cavity
  - cylinder of length L, radius R
- Standing wave modes
  - Calculation: Maxwell's equation $\rightarrow$ wave equation with metallic boundary conditions
  - Classification by number of nodes in r, ϕ, z
  - Best mode for acceleration: radial symmetric, max. E_z on axis
  - Other modes, higher order modes (HOM): beam instabilities !

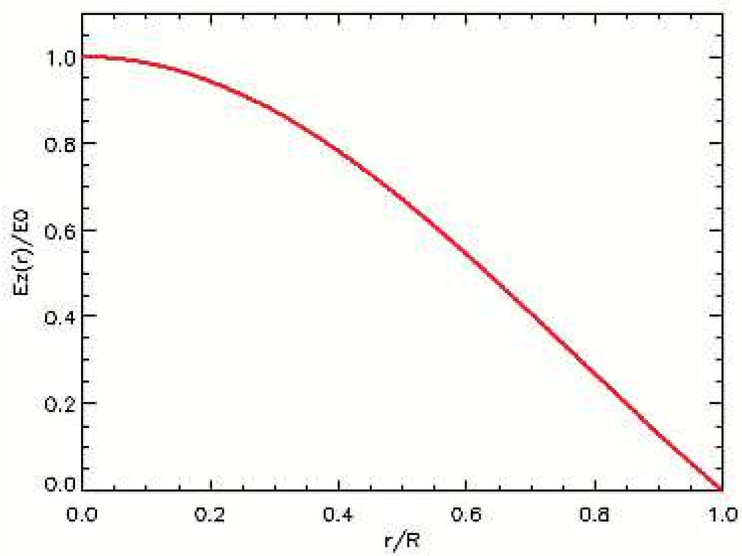
    $\Rightarrow$ HOM suppression = challenge in cavity design

◆ Longitudinal electric field in pillbox cavity

$$E_z(r,\phi,z,t) = E_0 \cdot J_m\left(y_{mn}\frac{r}{R}\right)\cdot \exp(\pm im\phi)\cdot \exp\left(i\pi p\frac{z}{L}\right)\cdot \exp(i\omega t)$$

- J_m is the m^{th} Bessel function and y_{mn} its n^{th} zero $(y_{01} = 2.405)$
- m, n, p are integers:
  - m azimuthal mode number, $m = 0$ radial symmetry.
  - n radial mode number
  - p longitudinal mode number, $p = 0$ longitudinally constant
- Mode definition: TM_{mnp} (TM = transverse magnetic)
- Accelerating mode: **TM_{010}**





Synchrotron oscillations

Circular machines (synchrotrons and storage rings):

- ▷ time dependant energy gain in RF cavity
- ▷ energy dependant time of recirculation

$\Longrightarrow$ *Synchrotron oscillation* of energy vs. time

Tune = number of oscillations per recirculation:

synchrotron tune Q_s, *betatron tunes* Q_x, Q_y

Synchrotrons: $Q_s \approx 10^{-4} \ldots 10^{-2} \ll Q_{x,y} \approx 1 \ldots 100$
(but e.g. microtron: $Q_s > Q_{x,y}$!)

$\Longrightarrow E(t)$, resp. $\delta(t) = \frac{\Delta p(t)}{p_o} = \frac{1}{\beta^2}\frac{\Delta E(t)}{E_o}$
treated as constant for betatron oscillations (adiabatic approximation)

Momentum compaction and transition

Time of flight $T = \frac{C(\delta)}{c\beta(\delta)}$, C = lattice circumference, $\delta = \frac{\Delta p}{p_o}$.

$$\frac{dT}{d\delta} = \frac{\partial T}{\partial C}\frac{dC}{d\delta} + \frac{\partial T}{\partial \beta}\frac{d\beta}{d\delta} = \frac{1}{c\beta_o}\frac{dC}{d\delta} - \frac{C_o}{c\beta_o^2}\frac{d\beta}{d\delta}$$

$\boxed{\frac{dC}{d\delta}}$

Dispersion: $x(s) = D(s)\cdot\delta \rightarrow$ path length $C(\delta) = \oint(\rho_o + D\delta)\,d\phi$

Dipole magnets: $d\phi = ds/\rho_o \quad \longrightarrow \quad C(\delta) = C_o + \oint \frac{D}{\rho_o}\,ds\cdot\delta$

$\left.\frac{dC}{d\delta}\right|_o = \alpha\, C_o$ with $\boxed{\alpha := \frac{1}{C_o}\oint\frac{D}{\rho_o}\,ds}$ the *momentum compaction factor*.

$\boxed{\frac{d\beta}{d\delta}}$

$\beta = \frac{\beta\gamma}{\sqrt{1+(\beta\gamma)^2}} \qquad p = m_o c\beta\gamma = p_o(1+\delta)$

$\beta = \left(1 + [\beta_o\gamma_o(1+\delta)]^{-2}\right)^{-1/2} \quad \longrightarrow \quad \left.\frac{d\beta}{d\delta}\right|_o = \frac{\beta_o}{\gamma_o^2}$

$$\Longrightarrow \quad \frac{dT}{d\delta} = \frac{C_o}{c\beta_o}\left(\alpha - \frac{1}{\gamma_o^2}\right) \quad \longrightarrow \quad \Delta T = T_o\eta\delta \text{ with } \eta := \alpha - \frac{1}{\gamma_o^2}$$

Competitive effects:

$\alpha \longrightarrow$ high momentum particles have longer path due to dispersion
$-1/\gamma_o^2 \longrightarrow$ high velocity particles are faster

Electron synchrotrons:

$\gamma_o \gg 1,\ \beta \approx 1 \quad \longrightarrow \quad \eta = \alpha$

Proton synchrotrons:

isochronous (i.e. $\eta = 0$) at *transition energy* $E_{\mathrm{tr}} = m_o c^2/\sqrt{\alpha}$

...use RF phase instead of time:

$\psi = 2\pi \frac{T}{\tau_{\mathrm{rf}}}$ with $\tau_{\mathrm{rf}} = \frac{1}{f_{\mathrm{rf}}} = \frac{T_o}{h}$, $h \in N$ the *harmonic number* and $T_o = \frac{C_o}{c\beta_o}$.

$\Delta\psi = 2\pi h \eta\, \delta$ or $\Delta\psi = \frac{2\pi h}{\beta_o^2} \eta \frac{\Delta E}{E_o}$

...use path length (longitudinal driftspace) instead of time:

$\Delta s = -\beta_o c \Delta T = -C_o \eta \delta$

Longitudinal phase space

Energy change of particle at phase ψ for one turn: $\Delta_t E(\psi) = qV \sin\psi - U(E)$

Note: $\Delta_t E = E(t + T_o) - E(t)$, but $\Delta E = E - E_o$. E_o = reference particle's energy

V = peak voltage of RF cavity. $U(E)$ = [energy dependant] energy loss per turn = $U_o + U'\Delta E$ with $U' = dU/dE$ (e.g. synchrotron radiation: $U \sim E^4$!).

Synchronous phase: $\Delta_t E = 0 = qV \sin\psi_s - U_o \quad \longrightarrow \quad \sin\psi_s = \frac{U_o}{qV}$

Use phase shift relative to synchronous particle $\Delta\psi = \psi - \psi_s$

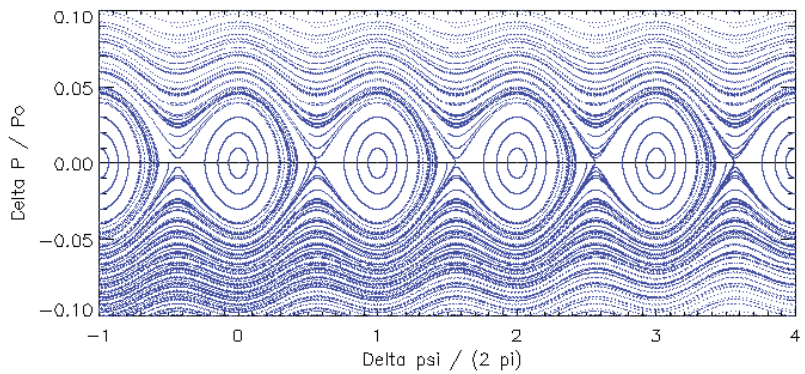
and relative momentum deviation $\delta = \frac{1}{\beta_o^2}\frac{\Delta E}{E_o}$

Synchrotron oscillation is "slow" $\rightarrow \dot{\delta} \approx \frac{\Delta_t \delta}{T_o}$ = change per turn.

$\Longrightarrow$ longitudinal non-linear equations of motion in δ and $\Delta\psi$:

$$\boxed{\dot{\delta} = \frac{qV}{\beta_o^2 E_o T_o} \cdot (\sin(\psi_s + \Delta\psi) - \sin\psi_s) - U'\frac{\delta}{T_o}}$$

$$\hookrightarrow \Delta\psi = 2\pi h\eta \cdot \delta \quad \longrightarrow \quad \boxed{(\dot{\Delta\psi}) = \frac{2\pi h\eta}{T_o}\,\delta}$$



The bucket

back to full non-linear equations of synchrotron motion:

$$\dot{\delta} = \frac{qV}{\beta_o^2 E_o T_o} \left(\sin(\psi_s + \Delta\psi) - \sin\psi_s \right) \qquad 2\delta = \frac{T_o}{\pi h \eta} (\dot{\Delta\psi}) \qquad [U' \approx 0]$$

$\rightarrow$ cross-wise multiplication, use $\frac{d}{dt}\delta^2 = 2\delta\dot{\delta}$ and $\frac{d}{dt}\cos(\psi_s + \Delta\psi) = -\sin(\psi_s + \Delta\psi)\,(\dot{\Delta\psi})$ and integrate $\rightarrow$

$$\delta^2 + \frac{qV}{\pi \beta_o^2 E_o h \eta} \underbrace{\left(\cos(\psi_s + \Delta\psi) + \sin\psi_s \, \Delta\psi \right)}_{:= W(\Delta\phi)} = \text{constant} := H$$

$\delta^2 \sim p^2 =$ kinetic energy, $W =$ potential, $H =$ Hamiltonian of the oscillation:

harmonic for small amplitudes: $\Delta\psi \ll 1 \quad \longrightarrow \quad \cos(\psi_s + \Delta\psi) \approx \cos\psi_s - \sin\psi_s \cdot \Delta\psi - \frac{1}{2}\cos\psi_s\,(\Delta\psi)^2$

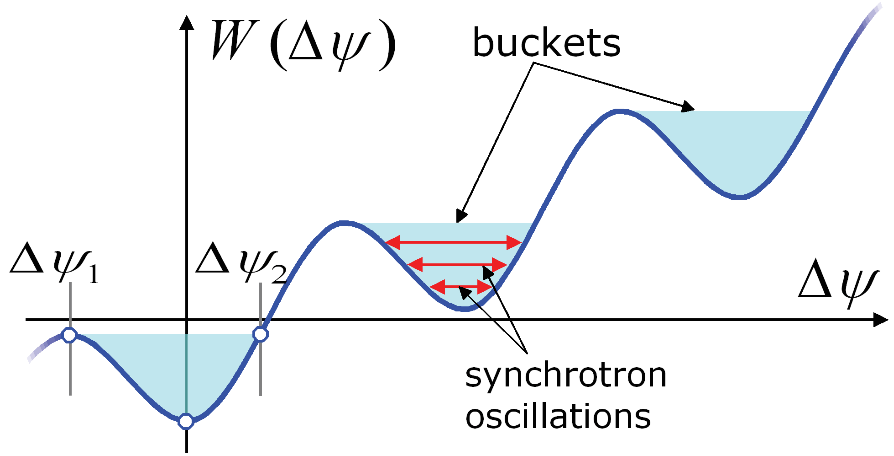
$\longrightarrow \delta^2 - k \cdot (\Delta\psi)^2 = H - 2k \qquad (k = \text{const.})$

$\longrightarrow$ ellipses in $(\Delta\psi, \delta)$ phase space for $\eta\cos\psi_s < 0$

Potential $W(\Delta\psi)$ for acceleration above transition $\rightarrow$

$\eta > 0$ and $90^\circ < \psi_s < 180^\circ$

$\longrightarrow \quad \sin\psi_s > 0, \cos\psi_s < 0.$



Formation of 2π-periodic regions of stable synchrotron oscillations, called *buckets*.

Phase acceptance

Separatrix = curve in phase space separating stable from unstable regions.

Phase acceptance = interval of stable phase $[\Delta\psi_1, \Delta\psi_2]$ for $\delta = 0$:

1. one limit given by bucket wall: $\frac{dW}{d\Delta\psi} \stackrel{!}{=} 0 \quad \longrightarrow \quad \sin(\psi_s + \Delta\psi) = \sin\psi_s$
   $\longrightarrow \quad \Delta\psi_o = 0$ (bucket bottom) and (since $\sin x = \sin(\pm\pi - x)$), $\Delta\psi_1 = \pm\pi - 2\psi_s$

2. other limit given by separatrix equipotential: $W(\Delta\psi_2) = W(\Delta\psi_1)$
   $\longrightarrow \quad$ solve numerically to get $\Delta\psi_2$. (for small buckets $\Delta\psi_2 \approx -\frac{1}{2}\Delta\psi_1$).

$\Longrightarrow$ Phase acceptance **above** (synchrotron) and **below** (linac) transition energy:

- $\psi_s \to 90°$:
  maximum acceleration: $\Delta_t E \to qV$
  minimum acceptance: $\Delta\psi_{1,2} \to 0$

- $\psi_s = 180°$ or $\psi_s = 0°$
  no acceleration: $\Delta_t E = 0$
  full acceptance: $\Delta\psi_{1,2} = \pm 180°$