A. Streun, Oct. 2017

Exercise 1

A. Lorentz force = centrifugal force $\quad \longrightarrow \quad evB = \frac{mv^2}{\rho}$

relativistically: $ec\beta B = \frac{m_o \gamma c^2 \beta^2}{\rho} \quad \longrightarrow \quad eB = \frac{m_o c \gamma \beta}{\rho} \quad \longrightarrow \quad (B\rho) = \frac{p}{e} = \frac{pc}{ec} = \frac{\beta E[\mathrm{eV}]}{c}$

This is just the magnetic rigidity from slide 69.

Momentum is not given in SI units of $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$ but in more convenient units of $\frac{\mathrm{eV}}{c}$:

$p\left[\frac{\mathrm{kg \cdot m}}{\mathrm{s}}\right] = \frac{e}{c} \cdot p\left[\frac{\mathrm{eV}}{c}\right]. \quad \longrightarrow \quad (B\rho) = \frac{1}{c} \cdot p\left[\frac{\mathrm{eV}}{c}\right].$

For $p = 7\,\mathrm{TeV}/c$ we get $(B\rho) = 23334\,\mathrm{Tm}$, and with $B = 8.33\,\mathrm{T}$ a bending radius of $2801\,\mathrm{m}$.

The total bending angle of a storage ring of course is 2π, so the total magnet length is $2\pi\rho$, and the length per bend $2\pi\rho/1232 = 14.3\,\mathrm{m}$.

B. magnet length $L = 18$ m (60% of circumference)

Magnetic rigidity:

$(B\rho) = B\frac{L}{2\pi} = 5$ T·m

$(B\rho) = \frac{p}{q} = \frac{m_o c \beta\gamma}{ne} = \frac{(m_o c^2/e)\beta\gamma}{nc}$

calculate $\beta\gamma \quad \longrightarrow \quad \beta, \gamma \quad \longrightarrow \quad T = m_o c^2(\gamma - 1)$

particle	q [e]	$m_o c^2$ [MeV]	$\beta\gamma$	γ	β	T [MeV]
positron	1	0.511	2943	2943	1	1500
proton	1	938.3	1.603	1.889	0.848	834
carbon ion	6	12 × 931.5	0.807	1.285	0.628	3190 or 266 MeV/u

Exercise 2

Lorentz force in electric field and relativistic momentum

$$F = \frac{dp}{dt} = qG \qquad p = m_o c\beta\gamma \qquad \longrightarrow \qquad d(\beta\gamma) = \underbrace{\frac{qG}{m_o c^2}}_{:=K} cdt$$

We are interested how the normalized momentum $(\beta\gamma)$ changes as time progresses in the moving system, where the lifetime is defined. The progression of time in the moving system is given by the Lorentz transformation. Since the moving particle stays at rest in its moving system, we have

$$c\,dt = \beta\gamma dz' + \gamma c\,dt' \quad \overset{z'=0}{\longrightarrow} \quad dt = \gamma\,dt'$$

For the change of momentum in the laboratory system as a function of time in the moving system we thus get

$$d(\beta\gamma) = K\gamma dt'$$

and with the identity $\gamma^2 = (\beta\gamma)^2 + 1$

$$\frac{d(\beta\gamma)}{\sqrt{(\beta\gamma)^2 + 1}} = Kc\,dt'$$

Integration gives the momentum as a function of time

$$\operatorname{arsinh}(\beta\gamma) = Kct' \qquad \longrightarrow \qquad (\beta\gamma) = \sinh(Kct')$$

The distance travelled is given by

$$dz = \beta cdt = \beta\gamma c\,dt'$$

We insert the result for $(\beta\gamma)$ and integrate over the length of the linac, where T' defines the time in the moving system when the linac end passes the particle (relativistically speaking...)

$$L = \int_0^L dz = \int_0^{T'} (\beta\gamma)c\,dt' = \frac{\cosh(KcT') - 1}{K} \qquad \longrightarrow \qquad T' = \frac{\operatorname{arcosh}(1 + KL)}{Kc},$$

and the percentage of surviving muons is $\exp(-T'/\tau)$.

Using again the identity $\gamma^2 = (\beta\gamma)^2 + 1$ we find

$$\gamma(t') = \sqrt{\beta\gamma)^2 + 1} = \sqrt{\sinh^2(Kct') + 1} = \cosh(Kct')$$

which gives the kinetic energy at the end of the linac:

$$E_{\text{kin}} = (\gamma(T') - 1)m_o c^2 = (\cosh(KcT') - 1)m_o c^2 = KLm_o c^2 = LqG$$

This is no surprise, since the energy gain corresponds to the integrated electric field (potential difference), which is just LG.

Finally, the time progressed in the laboratory system is given by

$$T = \int_0^{T'} \gamma dt' = \int_0^{T'} \cosh(Kct')dt' = \frac{\sinh(KcT')}{Kc} = \frac{p_{\text{end}}}{qG},$$

a relation which one could have derived immediately from the first equation, because Lorentz force was acting for time T, but we had to go this way to get explicit results for T and p_{end}.

Inserting numbers, we find that 91% of the muons reach the linac exit after 0.185 μs of their own time and 3.37 μs ($> \tau$!) in the laboratory. Kinetic energy is 10 GeV of course. For the pions it looks bad: only one out of 7400 would arrive at the linac end.

Exercise 3

Initial kinetic energy $T_o = 100$ keV, proton rest mass $m_o c^2 = 938.3$ MeV.

$\gamma = 1 + T_o/m_o c^2 = 1.00011 \quad \longrightarrow \quad \beta = 0.0146 \ll 1 \quad \longrightarrow$ sub-relativistic ok.

Assume start time delay Δt_o, this is the time when a proton is at the buncher cavity. If we assume that the cavity is "short", the kinetic energy gain in the cavity with max. voltage U_o is

$$\Delta T = eU_o \sin(\omega \Delta t_o) \;\; \text{with} \;\; \omega = 2\pi f \;\; \text{and} \;\; f = 50 \text{ MHz}$$

The time of arrival at the end of the driftspace L (where a linac would start) is given by

$$t = \Delta t_o + \frac{L}{v}$$

with the velocity v in sub-relativistic approximation given by

$$v = \sqrt{2\frac{T}{m_o}} = \sqrt{2\frac{(T_o + \Delta T)}{m_o}} = \underbrace{c \cdot \sqrt{2\frac{T_o}{m_o c^2}}}_{v_o} \cdot \sqrt{1 + \frac{eU_o}{T_o} \sin \omega \Delta t_o}$$

written conveniently this way since T_o is given in keV-units (not in Joule!).

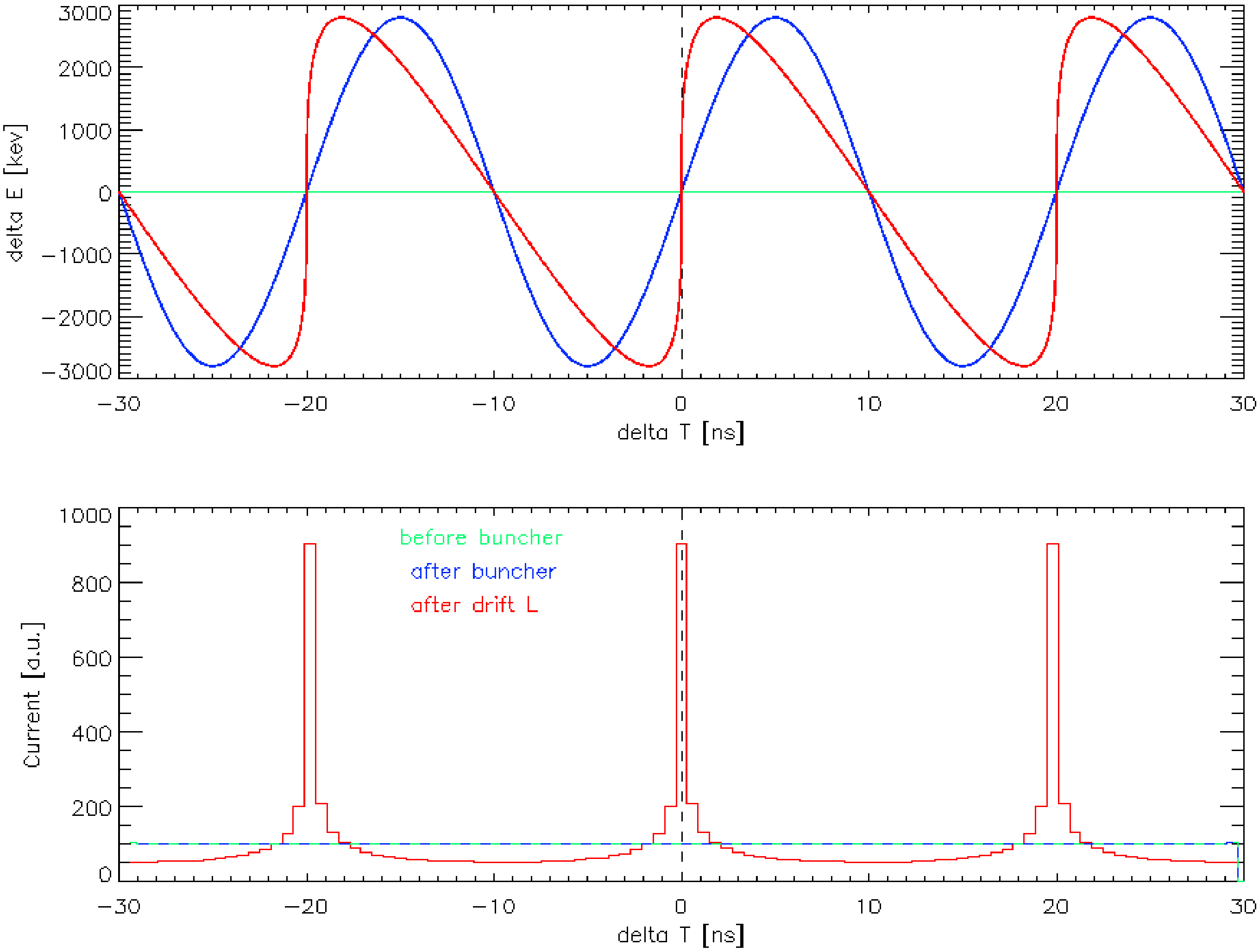
Optimum bunching is given when all particles arrive at the same time at the end of the drift space no matter what was the initial time Δt_o. Of course this works only for the linear part of the RF wave in the buncher cavity. One way would be to linearize the sin-function and proceed further, another, more elegant (but basically equivalent) way is differentiation: the arrival time has to be independant of the initial time, at least near our reference particle ($\Delta t_o = 0$):

$$\left.\frac{dt}{d\Delta t_o}\right|_{\Delta t_o = 0} \stackrel{!}{=} 0$$

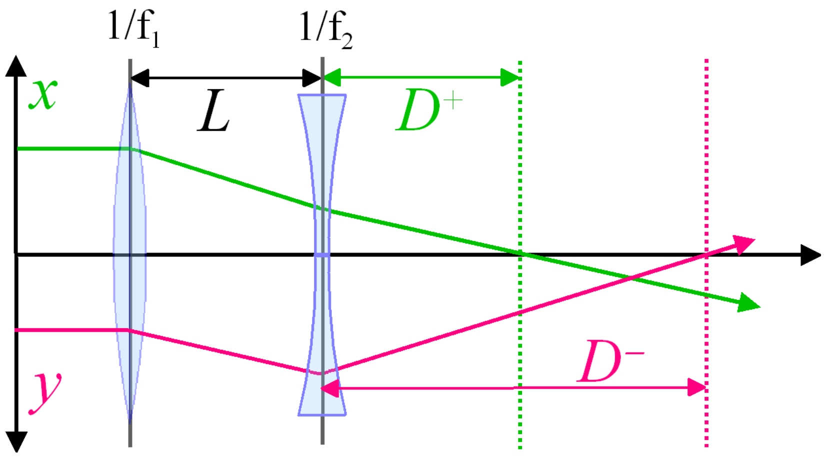
The rest is just algebra: insert t and v, differentiate with respect to Δt_o and then set $\Delta t_o = 0$. The result is

$$U_o = \frac{1}{\pi f L} \frac{c}{e} \sqrt{\frac{2T_o^3}{m_o c^2}} \;\; \text{and, with the given numbers,} \;\; U_o = 2788 \text{ V}.$$

The figure below shows the particle in longitudinal phase space relative to the reference particle, i.e. the coordinates are $\Delta t = t(\Delta t_o) - t(\Delta t_o = 0)$ and $\Delta T = T(\Delta t_o) - T(\Delta t_o = 0)$. The lower plots shows a histogram of the Δt distribution, i.e. the beam current. The buncher effect is well visible


3000
2000
1000
0
−1000
−2000
−3000
delta E [kev]
−30
−20
−10
0
10
20
30
delta T [ns]
1000
800
600
400
200
0
Current [a.u.]
before buncher
after buncher
after drift L
−30
−20
−10
0
10
20
30
delta T [ns]

Exercise 4



Abbreviations for convenience: $g = \frac{1}{f_1}$, $f = \frac{1}{f_2}$

Calculate the transfermatrix by multiplication of thin-lens- and drift-matrices:

$$M = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \mp f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \pm g & 1 \end{pmatrix}$$

The upper sign is for horizontal, the lower for vertical (or vice versa).

Multiplication gives

$$M = \begin{pmatrix} 1 \mp Df \pm Dg \pm Lg - LDfg & L + D \mp LDf \\ -Lgf \mp f \pm g & 1 \mp Lf \end{pmatrix}$$

A focus means, that an incoming particle with $\Delta x' = 0$ and arbitrary x is transformed to cross the axis, i.e. $x = 0$ and $\Delta x'$ irrelevant. Expressed as vector transformation ($*$ stands for any number):

$$\begin{pmatrix} 0 \\ * \end{pmatrix} = M \cdot \begin{pmatrix} * \\ 0 \end{pmatrix}$$

This requires that the upper left matrix element $m_{11} = 0$.

Solving $m_{11} = 0$ for the distance D to the focus gives two solutions (for horizontal and vertical):

$$D^{\pm} = \frac{1 \pm Lg}{Lfg \pm f \mp g}$$

We want the same focus distance in both dimensions: $D^+ = D^-$, i.e.

$$\frac{1 + Lg}{Lfg + f - g} \stackrel{!}{=} \frac{1 - Lg}{Lfg - f + g}$$

Solve this for f, the strength of the second lens which we want to know:

$$f = \frac{g}{1 - (gL)^2} \qquad \text{or} \qquad f_2 = f_1 \cdot \left(1 - \left(\frac{L}{f_1}\right)^2\right)$$

If this solution is introduced in the equation for $D^{\pm}$ we find, that all $\pm$ and $\mp$ symbols disappear as expected:

$$D = \frac{1 - (Lg)^2}{Lg^2} = \frac{1}{Lgf} \qquad \text{or} \qquad D = \frac{f_1 f_2}{L}$$

For the example shown we finally may confirm the numbers given.

Exercise 5

A. Calculate electric and magnetic fields as function of the particle position r inside the beam ($r < R$).

Use Maxwell equations in integral form, applying the theorems of Gauss and Stokes for trasnforming a volume integral over div$\vec{D}$ into a surface integral over $\vec{D}$, resp. the area integral over rot$\vec{H}$ into a line integral over $\vec{H}$:

$$\mathrm{div}\vec{D} = \rho \quad \longrightarrow \quad \int\int \vec{D} \cdot d\vec{\sigma} = \int\int\int \rho \, dV$$

$$\mathrm{rot}\vec{H} = \rho \quad \longrightarrow \quad \oint \vec{H} \cdot d\vec{s} = \int\int \vec{j} d\vec{\sigma}$$

Here ρ is the charge density in the beam, and $\vec{j} = \rho \, \vec{v}$ the corresponding current density of a beam propagating with velocity $\vec{v}$.

Apply the first equation to a thin slice of length Δz of the cylindrical beam. Due to symmetry the bottom and top of this slice will not contribute to the surface integral:

$$\int\int \vec{D} \cdot d\vec{\sigma} = D_r \, 2\pi r \, \Delta z = \rho \, \pi r^2 \, \Delta z \quad \longrightarrow \quad \vec{D} = \frac{\rho r}{2} \, \hat{\vec{e}}_r$$

For the magnetic field, we integrate along a circle of radius r:

$$\oint \vec{H} \cdot d\vec{s} = H_\phi \, 2\pi r = \vec{j} \, \pi r^2 \cdot \hat{\vec{e}}_z \quad \longrightarrow \quad \vec{H} = \frac{jr}{2} \, \hat{\vec{e}}_\phi$$

So the electric field is purely radial, the magnetic field purely azimuthal and both are linear in r (for $r < R$). Use $\epsilon_o \vec{E} = \vec{D}$, $\mu_o \vec{H} = \vec{B}$, $\vec{j} = \rho \, \vec{v}$ and $\epsilon_o \mu_o \, c^2 = 1$:

$$\longrightarrow \quad \vec{E} = \frac{\mu_o c^2 \rho}{2} \, r \, \hat{\vec{e}}_r \qquad \vec{B} = \frac{\mu_o v \rho}{2} \, r \, \hat{\vec{e}}_\phi$$

Now we calculate the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, first for the case of a particle travelling in the beam (case 1):

$$\vec{v} = v\hat{\vec{e}}_z \qquad q = +e \quad \longrightarrow \quad \vec{F} = e\frac{\mu_o c^2 \rho}{2} \, r \, \hat{\vec{e}}_r + e\frac{\mu_o v^2 \rho}{2} \, r \, \underbrace{\hat{\vec{e}}_z \times \hat{\vec{e}}_\phi}_{=-\hat{\vec{e}}_r}$$

$$\text{(case 1)} \qquad \vec{F} = \underbrace{\frac{e\mu_o c^2 \rho}{2}}_{:=\mathcal{F}} \, r \left(1 - \frac{v^2}{c^2}\right) \hat{\vec{e}}_r = \mathcal{F} \, r\hat{\vec{e}}_r (1 - \beta^2) \quad \xrightarrow{v \to c} \quad 0$$

Here we introduced the abbreviation $\mathcal{F}$ only for convenience. We see, that the space charge force is always defocusing, but disappears in the ultrarelativistic limit.

Now consider the two other cases:

$$\text{(case 2)} \qquad \vec{v} = -v\hat{\vec{e}}_z \qquad q = +e \quad \longrightarrow \quad \vec{F} = \mathcal{F} \, r\hat{\vec{e}}_r (1 + \beta^2) \quad \xrightarrow{v \to c} \quad 2\mathcal{F} \, r\hat{\vec{e}}_r$$

$$\text{(case 3)} \qquad \vec{v} = -v\hat{\vec{e}}_z \qquad q = -e \quad \longrightarrow \quad \vec{F} = -\mathcal{F} \, r\hat{\vec{e}}_r (1 + \beta^2) \quad \xrightarrow{v \to c} \quad -2\mathcal{F} \, r\hat{\vec{e}}_r$$

B. The focal length of a lens is geometrically given by the angle of deflection $\Delta r'$ of a particle that enters parallel to the z-axis at radius r: $\frac{1}{f} = \frac{\Delta r'}{r}$.

The deflection angle $\Delta r'$ is given by the angle between change of radial momentum and total momentum, $\Delta r' = \frac{\Delta p_r}{p}$, if we assume $\Delta r' \ll 1 \to p_z \approx p$.

The change of radial momentum is caused by the radial force:

$\dot{p}_r = F_r = -2\mathcal{F}\, r$ for case 3.

The total change of p_r after passing through the oncoming bunch of length L is

$$\Delta p_r = \int \dot{p}_r\, dt = -2\mathcal{F} \int r\, dt \approx -2\mathcal{F}\, r\, \Delta t$$

if we assume that r does not change much while passing through the bunch (thin lens approximation). The time of passage is given by $\Delta t = \frac{L}{2c}$ since the relative velocity of the particle and the oncoming bunch is $2c$ in the lab frame and in the ultrarelativistic limit.

$$\longrightarrow \quad \Delta p_r = -2\mathcal{F}\, r\, \frac{L}{2c} \rightarrow \Delta r' = -\frac{\mathcal{F} L}{pc}\, r = -\frac{\mathcal{F} L}{E}\, r,$$

since $pc = E$ in the ultrarelativistic limit. For the focal length we get

$$\frac{1}{f} = -\frac{\Delta r'}{r} = \frac{\mathcal{F} L}{E} = \frac{e \mu_o c^2 \rho}{2} \frac{L}{E}$$

The charge density of course is given by the number of particles N in the bunch divided by its volume (R is the envelope of the homogenously populated beam):

$$\rho = \frac{Ne}{\pi R^2 L}$$

We further introduce $E = m_o c^2 \gamma$ and the constant called classical electron radius $r_e = \frac{\mu_o e^2}{4\pi m_o}$ and finally get for the focal length:

$$\frac{1}{f} = \frac{e^2 \mu_o\, N}{2\pi m_o\, \gamma R^2} = \frac{2 r_e\, N}{\gamma R^2}$$

Put in the numbers $\quad \longrightarrow \quad f = 4.4$ cm.