SLS-Note 4/93
26.5.93

The influence of the electron beam's divergence on the photon beam's emittance in a synchrotron light source

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The effective emittances of photon beams from an undulator and from a superconducting bending magnet in the proposed "Swiss Synchrotron Light Source" were recently calculated by W. Joho [1]. It turned out, that with a bending magnet (at critical energy) the photon emittance can be significantly larger than the sum of the electron-emittance and the "diffraction emittance" (= phase space of a single radiating electron), whereas in a typical undulator the electron beam seemed to be well matched to diffraction. These results were obtained by using the convolution theorem for Gaussian distributions with <u>upright</u> ($\alpha \equiv 0$) phase-space-ellipses. In order to give a supplement to [1] we now will investigate the general case ($\alpha \neq 0$) and calculate the emittance of radiation from the <u>ends</u> of a magnetic device. Therefore we have to derive the convolution of 2-dimensional <u>correlated</u> Gaussian distributions

The phase-space ellipse for a particle beam is given by (Fig. 1):

(1) $\beta y'^2 + 2\alpha y y' + \gamma y^2 \qquad y = x_i z \qquad \beta\gamma = 1 + \alpha^2$

The transformation

(2) $$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

turns (1) into a circle (Fig. 2):

(3) $$\chi^2 + \eta^2 = \varepsilon$$

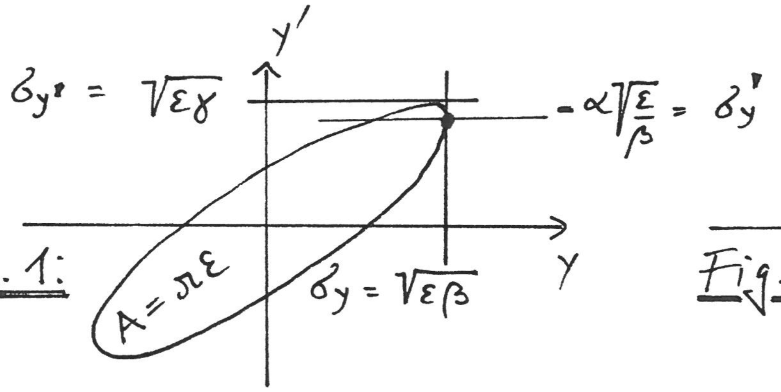


Fig. 1:

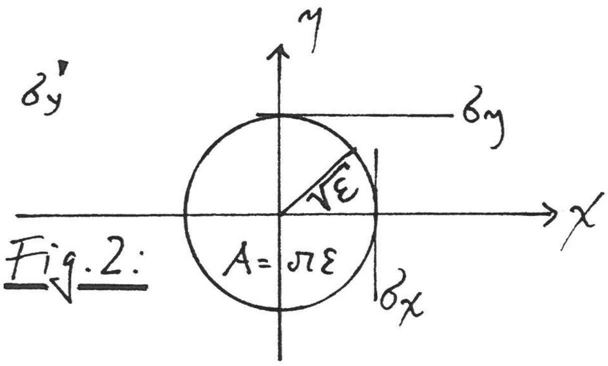


Fig. 2:

The 2-dimensional gaussian distribution for (3) is given by

(4) $$d^2N(\chi,\eta) = \frac{N_0}{\sqrt{2\pi}\,\sigma_\chi\sqrt{2\pi}\,\sigma_\eta} \cdot e^{-\frac{1}{2}\frac{\chi^2}{\sigma_\chi^2} - \frac{1}{2}\frac{\eta^2}{\sigma_\eta^2}}\, d\chi\, d\eta$$

Due to $\sigma_\chi = \sigma_\eta = \sqrt{\varepsilon}$ (Fig. 2) this may be written as

(5) $$d^2N(\chi,\eta) = \frac{N_0}{2\pi\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\chi^2 + \eta^2)}\, d\chi\, d\eta$$

The Jacobian of transformation (2) equals unity due to conservation of phase-space-area. Thus the correlated Gaussian particle-distribution in (y, y')-space is simply given by

$$d^2N(y,y') = \frac{N_0}{2\pi\varepsilon}\, e^{-\frac{1}{2\varepsilon}\left(\gamma y^2 + 2\alpha y y' + \beta y'^2\right)} \qquad (6)$$

We will now calculate the convolution of two distributions of type (6). The first distribution is given by

$$d^2N_1(y,y') = \frac{N_0}{2\pi\varepsilon_1}\, e^{-\frac{1}{2\varepsilon_1}\left(\gamma_1 y^2 + 2\alpha_1 y y' + \beta_1 y'^2\right)} \qquad (7)$$

The second distribution looks same in principle, but it is a distribution of every phase-space-element of the first one. (Fig 3.) Therefore we write

$$d^2N_2(u,u') = \frac{d^2N_1(y,y')}{2\pi\,\varepsilon_2}\, e^{-\frac{1}{2\varepsilon_2}\left(\gamma_2 u^2 + 2\alpha_2 u u' + \beta_2 u'^2\right)} \qquad (8)$$

with relative coordinates

$$u = \tilde{y} - y \qquad u' = \tilde{y}' - y'$$

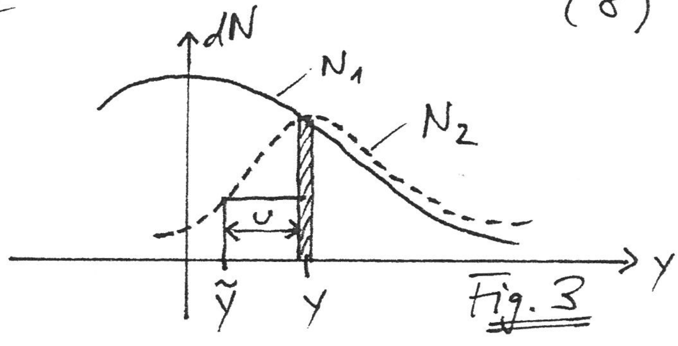


Fig. 3

To obtain the convolution, we integrate (8) in y and y' and get a new distribution in $\tilde{y}$ and $\tilde{y}'$:

$$(9)\quad d^2N(\tilde{y},\tilde{y}') = \frac{N_0}{4\pi^2\varepsilon_1\varepsilon_2}\iint_{-\infty}^{+\infty} e^{-\frac{1}{2\varepsilon_1}(\gamma_1 y^2 + 2\alpha_1 y y' + \beta_1 y'^2)} \cdot$$

$$\cdot\, e^{-\frac{1}{2\varepsilon_2}\left(\gamma_2(\tilde{y}-y)^2 + 2\alpha_2(\tilde{y}-y)(\tilde{y}'-y') + \beta_2(\tilde{y}'-y')^2\right)} dy\,dy'$$

Doing the integration is "straightforward but tedious". A lot of algebra and frequent use of the relation ($\beta\gamma = 1+\alpha^2$) eventually leads to a distribution of the convolution which looks like the distributions we had started with, Eq. (6):

$$(10)\quad d^2N(\tilde{y},\tilde{y}') = \frac{N_0}{2\pi\varepsilon}\, e^{-\frac{1}{2\varepsilon}(\gamma\tilde{y}^2 + 2\alpha\tilde{y}\tilde{y}' + \beta\tilde{y}'^2)} d\tilde{y}\,d\tilde{y}'$$

The emittance of the convoluted ellipse is

$$(11)\quad \varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_1\varepsilon_2(\beta_1\gamma_2 + \beta_2\gamma_1 - 2\alpha_1\alpha_2)$$

or alternatively written, defining $b =: \frac{\beta_1}{\beta_2}$

$$(11a)\quad \varepsilon^2 = (\varepsilon_1+\varepsilon_2)^2 + \varepsilon_1\varepsilon_2\left[\left(\sqrt{b} - \frac{1}{\sqrt{b}}\right)^2 + \left(\alpha_1\sqrt{b} - \alpha_2\cdot\frac{1}{\sqrt{b}}\right)^2\right]$$

For the special case of upright ellipses, we obtain the formula on page 3 of [1]:

$$\varepsilon^2 = (\varepsilon_1+\varepsilon_2)^2 + \varepsilon_1\varepsilon_2\left(\frac{1}{b} + b - 2\right)$$

The Twiss-parameters of the convoluted ellipse are given by

$$\gamma = \frac{\varepsilon_1\gamma_1 + \varepsilon_2\gamma_2}{\varepsilon} \qquad \beta = \frac{\varepsilon_1\beta_1 + \varepsilon_2\beta_2}{\varepsilon} \qquad \alpha = \frac{\varepsilon_1\alpha_1 + \varepsilon_2\alpha_2}{\varepsilon} \qquad (12)$$

We may alternatively express (12) with beam-envelope, angular envelope and divergence:

$$\delta_y = \sqrt{\varepsilon\beta} \qquad \delta_{y'} = \sqrt{\varepsilon\gamma} \qquad (\delta_y)' = -\alpha\sqrt{\frac{\varepsilon}{\beta}} \qquad \text{(Fig. 1)}$$

$$\Rightarrow \quad \delta_y^2 = \delta_{y_1}^2 + \delta_{y_2}^2 \qquad \delta_{y'}^2 = \delta_{y_1'}^2 + \delta_{y_2'}^2$$

$$\delta_y \delta_y' = \delta_{y_1}\delta_{y_1}' + \delta_{y_2}\delta_{y_2}' \qquad \left(\text{from (12) or } \frac{d}{ds}\delta_y^2\right) \qquad (13)$$

The well known convolution formula for upright ellipses apply to tilted ellipses too.

Using indices "e", "p", "d" for electron-beam, photon-beam and diffraction-ellipse, we may write Eq.(11a), using $\alpha_d = 0$, conveniently:

$$\varepsilon_p^2 = \varepsilon_e^2 + \varepsilon_d^2 + \varepsilon_e\varepsilon_d\left(b + \frac{1+\alpha_e^2}{b}\right) \qquad (14)$$

with $b = \frac{\beta_e}{\beta_d}$ as defined in [1].

We see from Eq.(14) that the electron beam's divergence ($\alpha_e \neq 0$) will have any influence only, if not $\beta_e \gg \beta_d$.

Calculation of photon-beam properties at center and at exit of magnetic devices in the SLS:

- The "K1"-lattice (OPTIK-File Q_OVA_K1) is used for the parameters of the storage ring. We assume:

  $E = 2.1$ GeV $\qquad \kappa = 10\%$

  $\Rightarrow \quad \varepsilon_{bx} = \underline{2.88 \text{ nm}} \qquad \varepsilon_{bz} = \underline{0.29 \text{ nm}}$

- We assume the same undulator as [1]:

  $L = 5$ m $\qquad \lambda_u = 5$ cm $\Rightarrow \lambda = 3$ nm

  Parameters of diffraction ellipse:

  $\beta_d = \frac{L}{4\pi} = \underline{0.4 \text{ m}} \qquad \varepsilon_d = \frac{\lambda}{4\pi} = \underline{0.25 \text{ nm}}$

  Parameters of electron beam:

  center: $\quad \beta_{ex} = \beta_{ez} = \underline{3.5 \text{ m}} \quad \alpha_{ex} = \alpha_{ez} = \underline{0}$

  exit: $\quad \beta_{ex} = \beta_{ez} = \underline{5.27 \text{ m}} \quad \alpha_{ex} = \alpha_{ez} = \underline{-0.714}$

- For the superconducting bending magnet, we take from the "K1"-lattice-data:

  $\rho = 1.5$ m $\qquad L = 0.26$ m $\qquad B = 4.67$ T

  Critical energy $\quad \epsilon_c = 665 \text{ eV} \cdot \frac{E^2}{\text{GeV}^2} \cdot \frac{B}{T} = 13.7$ keV

  $\Rightarrow \lambda_c = 0.09$ nm

  Diffraction ellipse: $\quad \varepsilon_d = \frac{\lambda}{4\pi} = \underline{7.2 \cdot 10^{-12} \text{ m}}$
  (at critical energy) $\quad \beta_d = \underline{2.9 \cdot 10^{-4} \text{ m}}$ [1]

  Parameters of the electron-beam:

  center: $\quad \beta_{ex} = 0.377$ m $\quad \beta_{ez} = 10.68$ m
  $\qquad \alpha_{ex} = 0 \quad \alpha_{ez} = 0$

  exit: $\quad \beta_{ex} = 0.419$ m $\quad \beta_{ez} = 10.68$ m
  $\qquad \alpha_{ex} = -0.346 \quad \alpha_{ez} = +0.615$

The results are shown in Fig. 4.–7.:

•) As found in [1] we see, that the matching of the electron-ellipse to the diffraction-ellipse is fair for the undulator and bad for the s.c. bend.

•) Further we see, that the divergence of the electron-beam seems to be negligible in most cases. This becomes quite clear when looking at Eq.(14): In most cases is valid: $\beta_e \gg \beta_d \Leftrightarrow b \gg 1$

•) The minimum emittance for the photon beam is given by (Eq.(14) and [1]):

$$\varepsilon_{p\,min} = \varepsilon_e + \varepsilon_d$$

for optimum matching ($\alpha_e = \alpha_d = 0$; $\beta_e = \beta_d$).

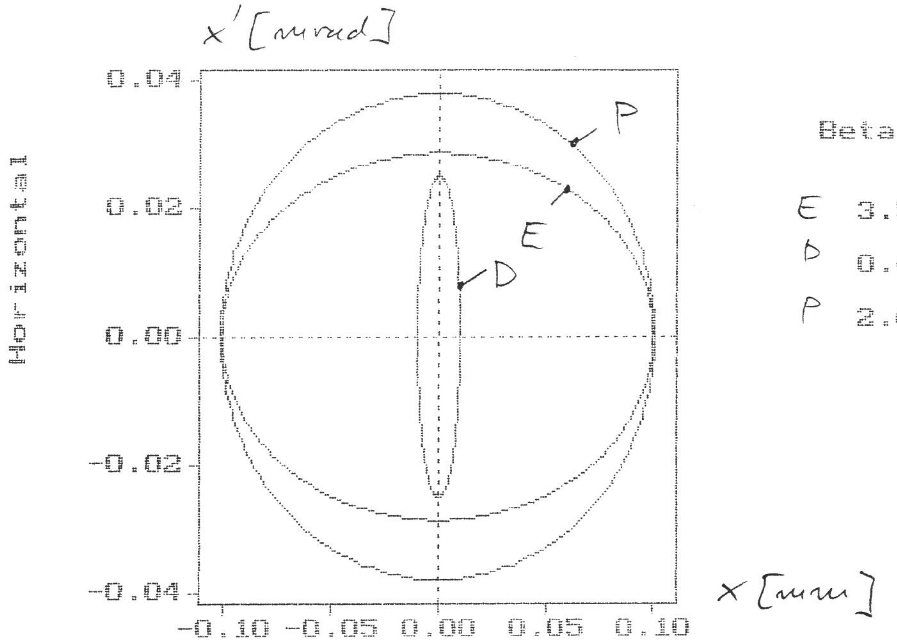
We have obtained, roughly, for the ratio $\varepsilon_p / \varepsilon_{p\,min}$ ("emittance dilution factor" [1])

		horizontal	vertical
- Undulator:			
	Center:	1.3	3.0
	Exit:	1.5	3.6
- s.c. bend:			
	Center:	2.1	30
	Exit:	2.2	30

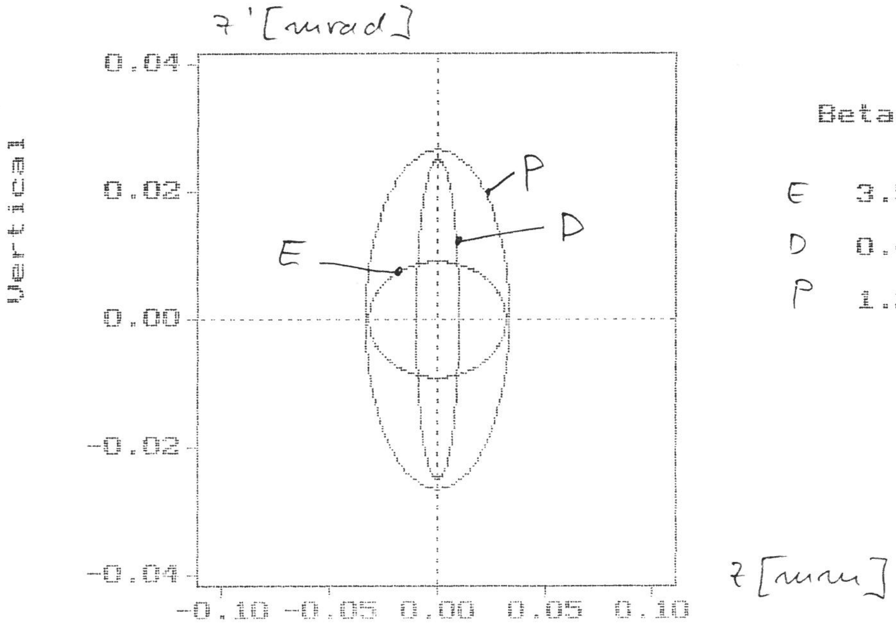
Note, that the differences between center and exit are mainly produced by the larger betafunction at the exit; the influence of $\alpha \neq 0$ is negligible.

Reference:

[1] W. Joho: "The effective emittance of a photon source" SLS-Note 3/93



	Beta [m]	Alpha	Emitt. [nm]
E	3.5000	0.0000	2.8800
D	0.4000	0.0000	0.2500
P	2.6516	0.0000	3.8392



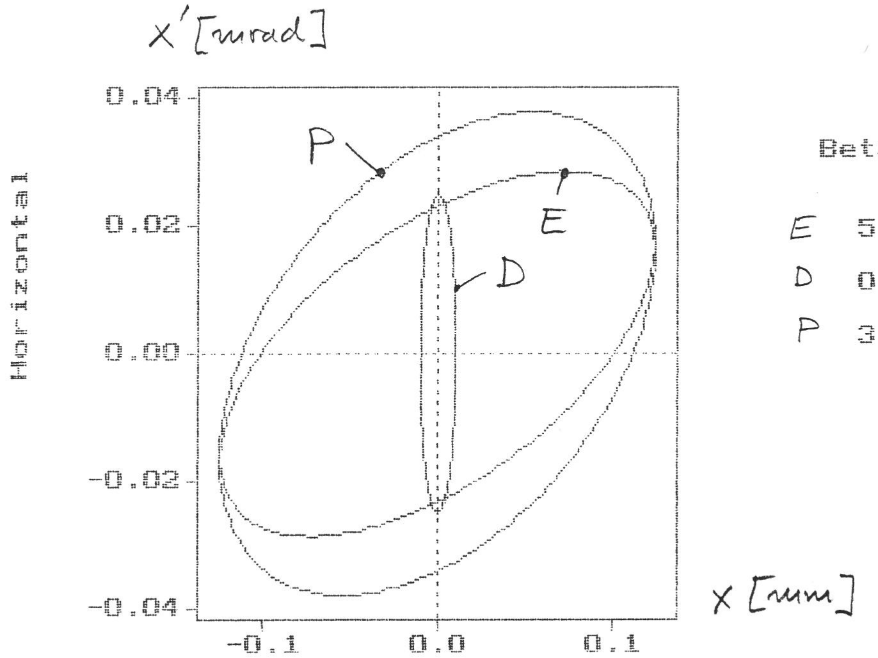
	Beta [m]	Alpha	Emitt. [nm]
E	3.5000	0.0000	0.2900
D	0.4000	0.0000	0.2500
P	1.2551	0.0000	0.8884

Fig. 4: Center of Undulator

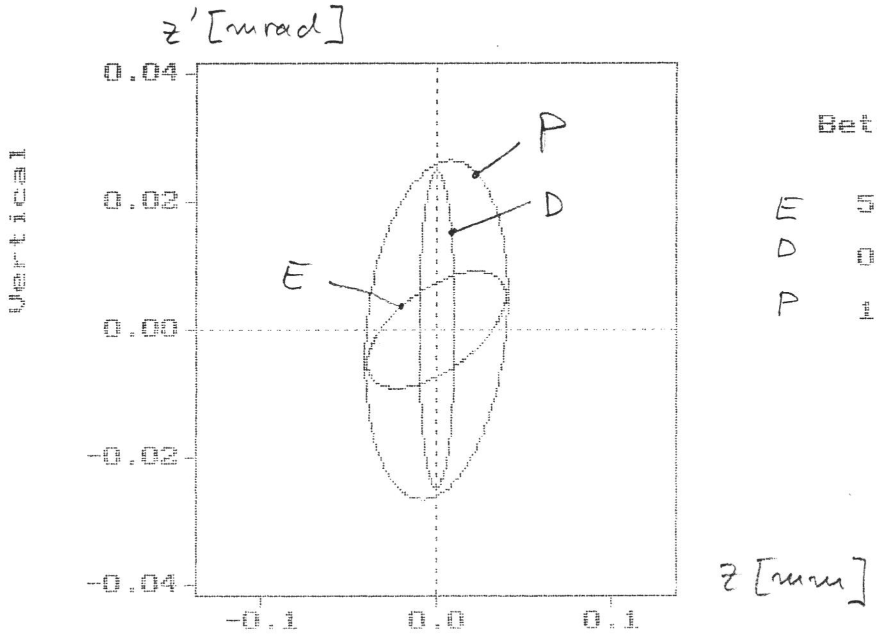
E : electron beam ellipse

D : diffraction ellipse

P : photon beam ellipse

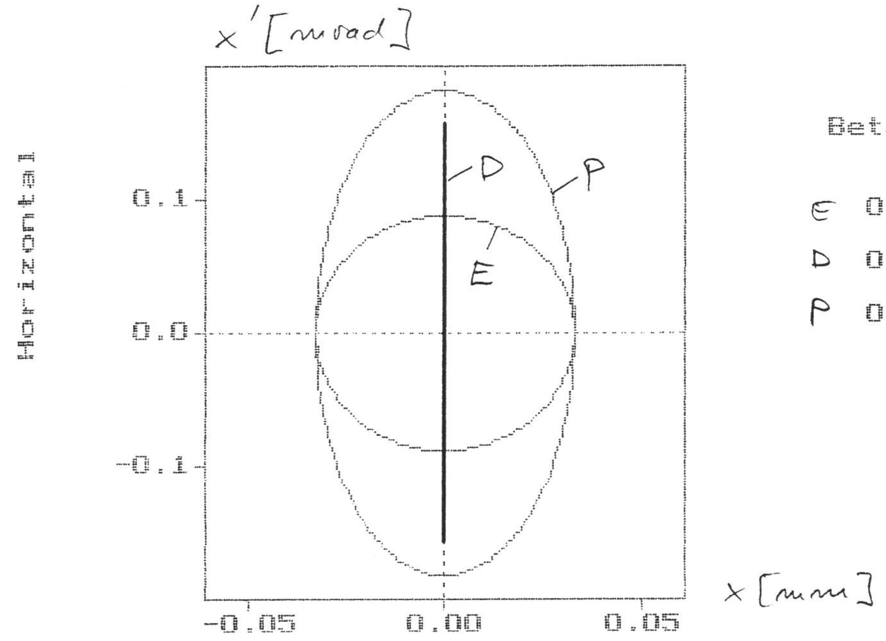


	Beta [m]	Alpha	Emitt. [nm]
E	5.2860	-0.7140	2.8800
D	0.4000	0.0000	0.2500
P	3.6165	-0.4853	4.2372

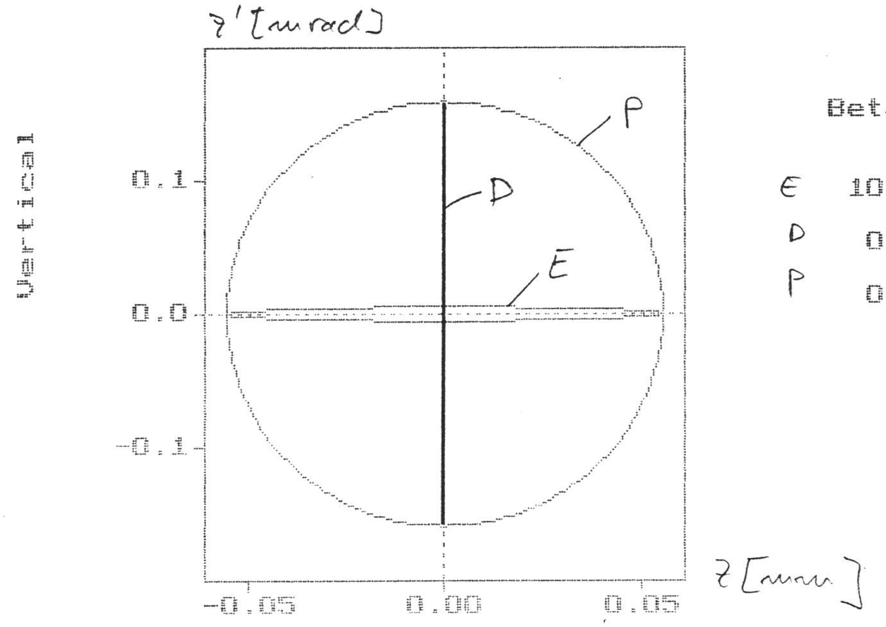


	Beta [m]	Alpha	Emitt. [nm]
E	5.2860	-0.7140	0.2900
D	0.4000	0.0000	0.2500
P	1.5478	-0.1963	1.0550

Fig. 5: End of Undulator



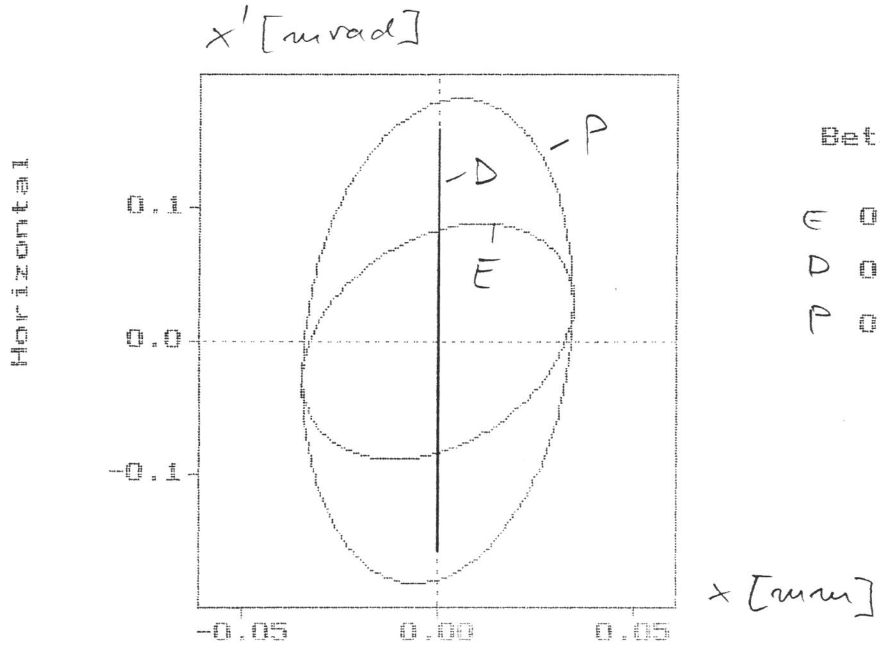
	Beta [m]	Alpha	Emitt. [nm]
E	0.3767	0.0000	2.8800
D	0.0003	0.0000	0.0072
P	0.1828	0.0000	5.9355



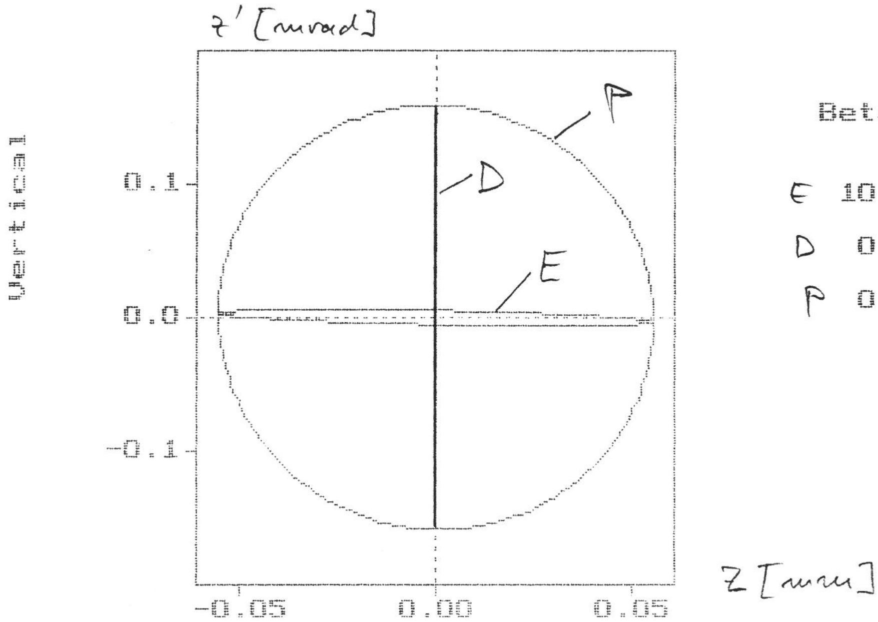
	Beta [m]	Alpha	Emitt. [nm]
E	10.6750	0.0000	0.2900
D	0.0003	0.0000	0.0072
P	0.3529	0.0000	8.7718

Fig. 6: Center of s.c. bend

E: electron beam ellipse
D: diffraction ellipse
P: photon beam ellipse



	Beta [m]	Alpha	Emitt. [nm]
E	0.4186	-0.3460	2.8800
D	0.0003	0.0000	0.0072
P	0.1950	-0.1612	6.1827



	Beta [m]	Alpha	Emitt. [nm]
E	10.6765	0.6150	0.2900
D	0.0003	0.0000	0.0072
P	0.3529	0.0203	8.7724

Fig. 7: End of s.c. bend